Impact of modifications on severity distributions

- A loss random variable $X$ may also be called **ground up loss**. This is the actual loss amount prior to any modification.

- The distribution of loss amount or of the cost to the insurer (claim amount) may be referred to as the **severity distribution**.

- If your house is insured for full value, then in case of any loss the insurer will bear the total loss. Therefore you may not be careful enough in preventing losses to happen. Because of these situations, the insurance companies may modify coverages. We study three types of adjustments in this course:

  1. deductible
     - ordinary deductible
     - franchise deductible
  2. policy limit
  3. coinsurance
Cost-Per-Loss Variable associated with ordinary deductibles

We first recall the notations:

\[ a \land b = \min(a, b) \quad a \lor b = \max(a, b) \]

Notation:

\[ x_+ = \begin{cases} 
  0 & x \leq 0 \\
  x & x > 0 
\end{cases} = \begin{cases} 
  0 & x < 0 \\
  x & x \geq 0 
\end{cases} \]

Especially:

\[ (x - d)_+ = \begin{cases} 
  0 & x \leq d \\
  x - d & x > d 
\end{cases} \]

The random variable

\[ Y^L = (X - d)_+ = \begin{cases} 
  0 & X \leq d \\
  X - d & X > d 
\end{cases} \]

is called the cost per loss variable.

This variable could also be called a left censored and shifted variable. It is called left censored because the values of \( X \) below \( d \) are set to zero. It is named shifted because \( d \) is subtracted from the remaining values.

The \( k \)-th moments:

\[
E((X - d)_+^k) = \int_d^\infty (x - d)^k f(x) dx \quad \text{for continuous } X
\]

\[
= \sum_{x_j > d} (x_j - d)^k p(x_j) \quad \text{for discrete } X
\]

The mean is

\[
E((X - d)_+) = \int_d^\infty (x - d) f(x) dx
\]

another formula for this is:

\[
E((X - d)_+) = \int_d^\infty S(x) dx
\]

2
Here is a proof: Since \( f(x) = -S'(x) \), we write:

\[
\int_d^\infty (x-d)f(x)dx = -\int_d^\infty (x-d)d\left(S(x)\right)
\]

\[
= -\left[(x-d)S(x)\right]_d^{+\infty} + \int_d^{+\infty} S(x)d(x-d)
\]

\[
= -\left[(x-d)S(x)\right]_d^{+\infty} + \int_d^{+\infty} S(x)dx
\]

\[
= 0 + \int_d^{+\infty} S(x)dx = \int_d^{+\infty} S(x)dx \quad \text{see the argument below}
\]

Here is a justification for the last equality in the above argument:

For \( x \geq d \) we have:

\[
0 \leq (x-d)S(x) = (x-d) \int_x^\infty f(t) \, dt = \int_x^\infty (x-d)f(t) \, dt \leq \int_x^{+\infty} (t-d)f(t) \, dt \quad (\star)
\]

But since the integral \( \int_0^\infty (t-d)f(t) \, dt = E(X-d) \) exists, the tail of the integral tends to zero, i.e. we have \( \lim_{x \to +\infty} \int_x^{+\infty} (t-d)f(t) \, dt = 0 \). Therefore the right-hand side of the inequality (\( \star \)) tends to zero, and therefore by an application of the Sandwich Theorem, we have \( \lim_{x \to +\infty} (x-d)S(x) = 0 \). This justifies the above argument.

Further note that

\[
\int_d^\infty (x-d)f(x)dx = \int_0^\infty (x-d)_+f(x)dx = E\left[(X-d)_+\right] = E\left[X - X \wedge d\right] = E\left[X\right] - E\left[X \wedge d\right]
\]

Therefore we can rewrite:

\[
E(Y^L) = E(X) - E(X \wedge d) = E((X-d)_+)
\]

Note.

Case 1. For \( x > 0 \) we have

\[
S_{Y^L}(x) = P(Y^L > x) = P(X - d > x) = P(X > x + d) = S_X(x + d)
\]
Upon differentiating we get $f_{Y^L}(x) = f_X(x + d)$ for $x > 0$

**Case 2.** For $x < 0$ we have

$$S_{Y^L}(x) = P(Y^L > x) = P(Y^L \geq 0) = 1 \quad \text{as } Y^L \geq 0$$

Upon differentiating we get $f_{Y^L}(x) = 0$ for $x < 0$

**Case 3.** For $x = 0$:

$$S_{Y^L}(0) = P(Y^L > 0) = P(X > d) = S_X(d)$$

But, for $f_{Y^L}$ at $x = 0$ we have:

$$f_{Y^L}(0) = P(Y^L = 0) = P(X \leq d) = F(d)$$

Here is a summary of these results:
\[
E(Y_L) = E(X) - E(X \wedge d) = E((X - d)_+) \quad (\ast)
\]

\[
E(Y^L) = \int_d^\infty S(x)dx \quad (\ast)
\]

\[
E[(X - d)^k_+] = \int_d^\infty (x - d)^k f(x)dx
\]

\[
f_{Y_L}(x) = \begin{cases} 
  f_X(x + d) & x > 0 \\
  F_X(d) & x = 0 \\
  0 & x < 0
\end{cases}
\]

\[
S_{Y_L}(x) = \begin{cases} 
  S_X(x + d) & x > 0 \\
  S_X(d) & x = 0 \\
  1 & x < 0
\end{cases}
\]

\[
h_{Y_L}(x) = \begin{cases} 
  \frac{f_X(x + d)}{S_X(x + d)} & x > 0 \\
  \frac{F_X(d)}{S_X(d)} & x = 0 \\
  0 & x < 0
\end{cases}
\]
Cost-Per-Payment (Excess Loss Variable) associated with ordinary deductibles

In the presence of an ordinary deductible only the losses above the threshold \( d \) are reported. In reality the conditional distribution \( (X - d \mid X > d) \) is the one that we deal with.

If \( P(X > d) > 0 \), then the random variable

\[
Y^P = (X - d \mid X > d) = \begin{cases} 
\text{undefined} & X \leq d \\
X - d & X > d 
\end{cases}
\]

is called the **excess loss** variable.

This variable could also be called a **left truncated and shifted** variable. It is called left truncated because the values of \( X \) below \( d \) are discarded. It is named shifted because \( d \) is subtracted from the remaining values.

**Mean excess loss function**:

\[
e_X(d) = e(d) = E(Y^P) = E(X - d \mid X > d)
\]

Other names for this mean are

\[
\begin{cases} 
\text{mean residual lifetime} \\
\text{complete expectation of life}
\end{cases}
\]

The \( k \)-th moments:

\[
e^k_X(d) = \begin{cases} 
\int_d^\infty (x - d)^k f(x) \, dx & \text{for continuous } X \\
\sum_{x_j > d} (x_j - d)^k p(x_j) / S(d) & \text{for discrete } X
\end{cases}
\]

Another formula for the mean excess function:

\[
e_X(d) = \frac{\int_d^\infty S(x) \, dx}{S(d)}
\]

The proof of this equality is the same as a similar formula seen in the previous section.
Further note that
\[
\int_d^\infty (x - d) f(x) dx = \int_0^\infty (x - d)_+ f(x) dx = E[(X - d)_+] = E[X - X \wedge d] = E[X] - E[X \wedge d]
\]

Therefore we can rewrite:
\[
E(Y^P) = \frac{E(X) - E(X \wedge d)}{S_X(d)} = \frac{E((X-d)_+)}{S_X(d)}
\]

Note.

**Case 1.** For \(x > 0\) we have

\[
S_{Y^P}(x) = P(Y^P > x) = P(X - d > x \mid X > d) = P(X > x + d \mid X > d)
\]

\[
= \frac{P(X > x + d, X > d)}{P(X > d)} = \frac{P(X > x + d)}{P(X > d)} = \frac{S_X(x + d)}{S_X(d)}
\]

Upon differentiating we get \(f_{Y^L}(x) = \frac{f_X(x+d)}{S_X(d)}\) for \(x > 0\)

**Case 2.** For \(x < 0\) we have

\[
S_{Y^P}(x) = P(Y^P > x) = P(X - d > x \mid X > d) = P(X > x + d \mid X > d)
\]

\[
= \frac{P(X > x + d, X > d)}{P(X > d)} = \frac{P(X > d)}{P(X > d)} = 1
\]

Upon differentiating we get \(f_{Y^L}(x) = 0\) for \(x < 0\)

**Case 3.** For \(x = 0\):

\[
S_{Y^P}(0) = P(Y^P > 0) = P(X - d > 0 \mid X > d) = P(X > d \mid X > d) = 1
\]

and:

\[
f_{Y^P}(0) = P(Y^P = 0) = P(X - d = 0 \mid X > d) = P(X = d \mid X > d) = 0
\]

Here is a summary of these results:
\[
\begin{align*}
    e(d) &= \frac{E(X) - E(X^d)}{S_X(d)} = \frac{E((X-d)_+)}{S_X(d)} \\
    e(d) &= \frac{\int_d^\infty S(x)dx}{S(d)} \\
    e^k(d) &= \frac{\int_d^\infty (x-d)^k f(x)dx}{S(d)} \\
    f_{Y_{FP}}(x) &= \begin{cases} 
        \frac{f_X(x+d)}{S_X(d)} & x > 0 \\
        0 & x \leq 0 
    \end{cases} \\
    S_{Y_{FP}}(x) &= \begin{cases} 
        \frac{S_X(x+d)}{S_X(d)} & x > 0 \\
        1 & x \leq 0 
    \end{cases} \\
    h_{Y_{FP}}(x) &= \begin{cases} 
        \frac{f_X(x+d)}{S_X(x+d)} & x > 0 \\
        1 & x \leq 0 
    \end{cases}
\end{align*}
\]

**Example.** Calculate \( e(d) \) for Exponential(\( \theta \))

**Solution.** From the Exam C table we have \( S(x) = e^{-\frac{x}{\theta}} \).

\[
\int_d^\infty S(x) \, dx = \int_d^\infty e^{-\frac{x}{\theta}} \, dx = [\theta e^{-\frac{x}{\theta}}]_d^\infty = \theta e^{-\frac{d}{\theta}}
\]

And according to the table, we have:

\( S(d) = e^{-\frac{d}{\theta}} \)

Then:

\[
e(d) = \frac{\int_d^\infty S(x) \, dx}{S(d)} = \theta \quad \checkmark
\]

**Example.** Calculate \( e(d) \) for Pareto(\( \alpha, \theta \)) with \( \alpha > 1 \)
Solution. From the Exam C table we have \( S(x) = \left( \frac{\theta}{x+\theta} \right)^\alpha \).

\[
\int_d^\infty S(x) \, dx = \theta^\alpha \int_d^\infty (x + \theta)^{-\alpha} \, dx = \theta^\alpha \left[ \frac{(x + \theta)^{-\alpha + 1}}{-\alpha + 1} \right]_d^\infty = \frac{\theta^\alpha}{1 - \alpha} \left[ \frac{1}{(x + \theta)^{\alpha - 1}} \right]_d^\infty
\]

\[
= \left( \frac{\theta^\alpha}{\alpha - 1} \right) \left( \frac{1}{(d + \theta)^{\alpha - 1}} \right)
\]

And according to the table, we have:

\[
S(d) = \left( \frac{\theta}{d + \theta} \right)^\alpha
\]

Then:

\[
e(d) = \frac{\int_d^\infty S(x) \, dx}{S(d)} = \frac{d + \theta}{\alpha - 1} \quad \checkmark
\]
Cost-Per-Loss associated with franchise deductibles

With a franchise deductible, the loss is fully paid when it is over the threshold.

The associated cost-per-loss random variable is defined by:

\[ Y^L = \begin{cases} 
0 & X \leq d \\
X & X > d 
\end{cases} \]

Here are the related functions:

\[ f_{Y^L}(x) = \begin{cases} 
F_X(d) & x = 0 \\
f_X(x) & x > d 
\end{cases} \]

\[ S_{Y^L}(x) = \begin{cases} 
S_X(d) & 0 \leq x \leq d \\
S_X(x) & x > d 
\end{cases} \]

\[ h_{Y^L}(x) = \begin{cases} 
0 & 0 < x < d \\
h_X(x) & x > d 
\end{cases} \]

\[ E(Y^L) = E(X) - E(X \wedge d) + d S_X(d) \quad (\star) \]

**Proof of this last equality:** The random variable \( Y^L \) takes on the value zero with probability \( F_X(d) \). Their product is zero. The other points that we should consider are the points \( x > d \). Therefore:

\[ E(Y^L) = \int_d^\infty x f_X(x) \, dx - \int_d^{d} (x-d) f_X(x) \, dx + d \int_d^\infty f_X(x) \, dx = E[(X-d)_+] + d S_X(d) \]
Cost-Per-Payment associated with franchise deductibles

For a franchise deductible, the associated cost-per-payment random variable is defined by:

\[ Y^P = \begin{cases} \text{undefined} & X \leq d \\ X & X > d \end{cases} \]

Here are the related functions:

\[ f_{Y^P}(x) = \begin{cases} f_X(x) & x > d \end{cases} \]

\[ S_{Y^P}(x) = \begin{cases} 1 & 0 \leq x \leq d \\ \frac{S_X(x)}{S_X(d)} & x > d \end{cases} \]

\[ h_{Y^P}(x) = \begin{cases} 0 & 0 < x < d \\ h_X(x) & x > d \end{cases} \]

\[ E(Y^P) = \frac{E(X) - E(X \wedge d)}{S_X(d)} + d \quad (\ast) \]

Proof.

\[ E(Y^P) = \frac{E(Y^L)}{S_X(d)} = \frac{E(X) - E(X \wedge d)}{S_X(d)} + d \]

Note. This value is equal to \( d \) plus the mean excess loss.

Example. Auto liability losses for a group of insureds (Group R) follow a Pareto distribution with \( \alpha = 2 \) and \( \theta = 2,000 \). Losses from a second group (Group S) follow a Pareto distribution with \( \alpha = 2 \) and \( \theta = 3,000 \). Group R has an ordinary deductible of 500, while Group S has a franchise deductible of 200. Calculate the amount that the expected cost per payment for Group S exceeds that for Group R.

Solution. For group R we have the expected value equal to:
\[ E_R = \frac{d + \theta}{\alpha - 1} = \frac{500 + 2000}{2 - 1} = 2500 \]

For group \( S \):

\[ E_S = \text{expectation associated with a ordinary deductible} + d = \frac{d + \theta}{\alpha - 1} + d = \frac{200 + 3000}{2 - 1} + 200 = 3400 \]

\[ E_S - E_R = 3400 - 2500 = 900 \]
Loss Elimination Ratio

Consider a cost-per-loss contract. On average, the insurance company pays the amount $E[(X - d)_+]$, so it saves the amount $E(X) - E[(X - d)_+]$. The ratio

$$\frac{E(X) - E[(X - d)_+]}{E(X)} = \frac{E(X \wedge d)}{E(X)} = \frac{\int_0^d S(x) \, dx}{\int_0^\infty S(x) \, dx}$$

gives the percentage of the expected loss that the insurer saves. This number is called the loss elimination ratio.

Example (from the textbook). Determine the loss elimination ratio for the Pareto distribution with $\alpha = 3$ and $\theta = 2,000$ with an ordinary deductible of 500, and interpret this number.

Solution. Somewhere we calculated that

$$E[(X - d)_+] = \left( \frac{\theta^\alpha}{\alpha - 1} \right) \left( \frac{1}{(d + \theta)^{\alpha-1}} \right)$$

On the other hand,

$$E(X) = \frac{\theta}{\alpha - 1}$$

Therefore

$$\frac{E[(X - d)_+]}{E(X)} = \left( \frac{\theta}{(d + \theta)} \right)^{\alpha-1} = \left( \frac{2000}{(500 + 2000)} \right)^2 = 0.64.$$ 

Then

$$LER = 1 - \frac{E[(X - d)_+]}{E(X)} = 0.36$$

This means that by introducing an ordinary deductible of 500 we can eliminate 36% of losses.

Example (Exercise 8.10 textbook). Losses have an exponential distribution with a mean of 1,000. There is a deductible of 500. Determine the amount by which the deductible would have to be raised to double the loss elimination ratio.

Solution.
Step 1. For any deductible $d$ we have
\[ \int_0^d S(x)dx = \int_0^d e^{-\frac{x}{1000}}dx = \left[-1000e^{-\frac{x}{1000}}\right]_0^d = 1000 \left(1 - e^{-\frac{d}{1000}}\right) \]
By letting $d \to \infty$ we get:
\[ \int_0^\infty S(x)dx = 1000 \quad \text{which we already know for exponential distribution.} \]
Then:
\[ LER = \frac{\int_0^d S(x)}{\int_0^\infty S(x)dx} = 1 - e^{-\frac{d}{1000}} \]

For the deductible $d = 500$ this becomes $1 - e^{-0.5}$. We need to find $d$ so as to have $1 - e^{-\frac{d}{1000}} = 2(1 - e^{-0.5}) = 0.3935$.

By using the logarithmic function we get $d = 1546$.

Example (Exercise 8.13 textbook) *. Losses have a mean of 2,000. With a deductible of 1,000, the loss elimination ratio is 0.3. The probability of a loss being greater than 1,000 is 0.4. Determine the average size of a loss given it is less than or equal to 1,000

Solution.

In general:
\[ E(X|A) = \frac{1}{P(A)} \int_A x f_X(x) dx \]

In particular:
\[ E(X|X \leq 1000) = \frac{1}{P(X \leq 1000)} \int_0^{1000} x f_X(x) dx = \frac{1}{0.6} \int_0^{1000} x f_X(x) dx \quad (*) \]

So we need to find $\int_0^{1000} x f_X(x) dx$.

But note that:
\[ E(X \land 1000) = \int_0^\infty (x \land 1000) f_X(x)dx = \int_0^{1000} x f_X(x)dx + \int_{1000}^\infty 1000 f_X(x)dx = \]

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\[
\int_0^{1000} x f_X(x) \, dx + 1000 \int_0^\infty f_X(x) \, dx = \int_0^{1000} x f_X(x) \, dx + 1000 \, P(X > 1000)
\]

\[
= \int_0^{1000} x f_X(x) \, dx + 400
\]

\Rightarrow \quad E(X \wedge 1000) = \int_0^{1000} x f_X(x) \, dx + 400 \quad (**)

Next

\[
LER = \frac{E(X \wedge 1000)}{E(X)} \quad \Rightarrow \quad 0.3 = \frac{E(X \wedge 1000)}{200} \quad \Rightarrow \quad E(X \wedge 1000) = 600 \quad (**)
\]

\[
\int_0^{1000} x f_X(x) \, dx = 200 \quad (*) \quad E(X|X \leq 1000) = \frac{200}{0.6} = 333
\]

**Example (from the Finan’s notes).** An insurance company offers two types of policies: Type A and Type B. The distribution of each type is presented below.

<table>
<thead>
<tr>
<th>Type A</th>
<th>Type B</th>
</tr>
</thead>
<tbody>
<tr>
<td>loss amount</td>
<td>loss amount</td>
</tr>
<tr>
<td>probability</td>
<td>probability</td>
</tr>
<tr>
<td>100</td>
<td>300</td>
</tr>
<tr>
<td>0.65</td>
<td>0.70</td>
</tr>
<tr>
<td>200</td>
<td>400</td>
</tr>
<tr>
<td>0.35</td>
<td>0.30</td>
</tr>
</tbody>
</table>

55% of the policies are of Type A and the rest are of Type B. For an ordinary deductible of 125, calculate the loss elimination ratio and interpret its value.

**Solution.**

The expected loss:

\[
E(X) = E(X|A)P(A) + E(X|B)P(B) =
\]

\[
[(100)(0.65) + (200)(0.35)(0.55) + [(300)(0.70) + (400)(0.30)](0.45)] = 222.75
\]

The expected savings:

\[
E(X \wedge 125) = E(X \wedge 125|A)P(A) + E(X \wedge 125|B)P(B) =
\]

\[
[(100)(0.65) + (125)(0.35)(0.55) + (125)(0.45)] = 116.0625
\]
Loss elimination ratio:

\[ \text{LER} = \frac{116.0625}{222.75} = 0.521 = 52.1\% \]
Effect of Inflation on Average Cost-Per-Loss and Average Cost-Per-Payment

In the presence of inflation, costs will increase but the deductibles are unchanged. If \( X \) is the original ground-up variable, the inflated loss variable will be \((1 + r)X\). In case of a deductible \( d \), a payment is going to occur if and only if \((1 + r)X > d\) equivalently \( X > \frac{d}{1+r} \). Because of this, the associated per-loss variable is

\[
Y^L = \begin{cases} 
0 & X \leq \frac{d}{1+r} \\
(1 + r)X - d & X > \frac{d}{1+r}
\end{cases}
\]

\[
= (1 + r) \begin{cases} 
0 & X \leq \frac{d}{1+r} \\
X - \frac{d}{1+r} & X > \frac{d}{1+r}
\end{cases}
\]

\[
= (1 + r) \left[ X - X \wedge \frac{d}{1+r} \right]
\]

The associated per-payment variable is

\[
Y^P = \begin{cases} 
\text{undefined} & X \leq \frac{d}{1+r} \\
(1 + r)X - d & X > \frac{d}{1+r}
\end{cases}
\]

**Theorem.** Let \( X \) be the loss random variable and let \((1 + r)X\) be the loss random variable after a uniform inflation of \( r \). Then, for an ordinary deductible of \( d \) we have

\[
E(Y^L) = (1 + r) \left[ E(X) - E \left( X \wedge \frac{d}{1+r} \right) \right]
\]

\[
E(Y^P) = \frac{1}{S_X \left( \frac{d}{1+r} \right)} (1 + r) \left[ E(X) - E \left( X \wedge \frac{d}{1+r} \right) \right]
\]

In other words, the process for calculating \( E(Y^L) \) and \( E(Y^P) \) is this:
**First:** change the deductible to \( \frac{d}{1+r} \)

**Second:** Calculate the average cost-per-loss and average cost-per-payment for this new deductible and the old \( X \).

**Third:** Multiply the results by \( (1 + r) \). The numbers such found are the average cost-per-loss and average cost-per-payment associated with the inflated \( (1 + r)X \).

**Example (exercise 8.11 of the textbook)**. The values in the following table are available for a random variable \( X \). There is a deductible of 15,000 per loss and no policy limit. Determine the expected cost per payment using \( X \) and then assuming 50% inflation (with the deductible remaining at 15,000).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( F(x) )</th>
<th>( E(X \wedge x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10,000</td>
<td>0.60</td>
<td>6,000</td>
</tr>
<tr>
<td>15,000</td>
<td>0.70</td>
<td>7,700</td>
</tr>
<tr>
<td>22,500</td>
<td>0.80</td>
<td>9,500</td>
</tr>
<tr>
<td>( \infty )</td>
<td>1.00</td>
<td>20,000</td>
</tr>
</tbody>
</table>

**Solution.** Before inflation:

\[
\frac{E(X) - E(X \wedge 15000)}{S(15000)} = \frac{20000 - 7700}{1 - 0.70} = 41000
\]

After inflation:

First change the deductible to \( \frac{15000}{1.5} = 10000 \)

Then calculate the expected cost per payment for \( X \) with this new deductible:

\[
\frac{E(X) - E(X \wedge 10000)}{S(10000)} = \frac{20000 - 6000}{1 - 0.60} = 35000
\]

Then multiply by 1.5 to get the expected cost per payment after inflation:

**Expected cost per payment after inflation** = \((1.5)(35000) = 52500\)
Policy Limits

In this type of contracts the insurer compensates the insured up to a policy limit \( u \). If \( X \) is the ground up loss, then the random variable that reflects this policy is the **limited-loss variable** defined by:

\[
Y = X \wedge u = \begin{cases} 
X & X \leq u \\
u & X > u
\end{cases}
\]

The expected value \( E(X \wedge u) \) is called **limited expected value**.

**Theorem.** For the limited loss variable \( Y \) we have

\[
E(Y) = \int_0^u S(x)dx
\]

**Proof.**

\[
Y = X + \begin{cases} 
0 & X \leq u \\
u - X & X > u
\end{cases}
\]

\[
= X - \begin{cases} 
0 & X \leq u \\
X - u & X > u
\end{cases}
\]

So \( Y = X - Y^L \) for where \( u \) for \( Y^L \) plays the role of an ordinary deductible. Therefore:

\[
E(Y) = E(X) - E(Y^L) = \int_0^\infty S(x)dx - \int_u^\infty S(x)dx = \int_0^u S(x)dx
\]

**Theorem.** For positive values \( u \) and \( r \) we have

\[
E\left((1 + r)X \wedge u\right) = (1 + r)E\left(X \wedge \frac{u}{1 + r}\right)
\]

**Note.** If \( r \) is a uniform inflation rate, then this equality means that the expected value of the limited loss variable after inflation (the left-hand side of the equality) is found by
finding the mean limited loss for $\frac{u}{1+r}$ and then multiplying it by $1+r$. So it is basically the same three-steps method we have had for the impact of inflation on $Y^L$ and $Y^P$ for ordinary deductibles.

**Example (from the Finan’s note).** Losses follow a Pareto distribution with parameters $\alpha = 2$ and $\theta = 1000$. For a coverage with policy limit 2000 and after an inflation rate of 30%, calculate the after inflation expected cost.

**Solution.**

From the table at the end of the textbook, for $a \neq 1$ we have

$$E(X \wedge u) = \frac{\theta}{\alpha - 1} \left[ 1 - \left( \frac{\theta}{u + \theta} \right)^{\alpha - 1} \right].$$

**Step 1** We change $u = 2000$ to $\frac{u}{1+r} = \frac{2000}{1.3}$.

**Step 2** We calculate $E \left( X \wedge \frac{u}{1+r} \right) = E \left( X \wedge \frac{2000}{1.3} \right) = \frac{1000}{2-1} \left[ 1 - \left( \frac{1000}{\frac{2000}{1.3}+1000} \right) \right] = 606.06$

**Step 3** Multiply it by 1.3: Answer = (1.3)(606.06) = 787.88

**Example.** You are given the following: The underlying distribution for 1993 losses is given by $f(x) = e^{-x}$, $x > 0$, where losses are expressed in millions of dollars.

Inflation of 5% impacts all claims uniformly from 1993 to I994.

Under a basic limits policy, individual losses are capped at 1.0 million in each year. What is the inflation rate from 1993 to 1994 on the capped losses?

**Solution.** If $X_{1993}$ and $X_{1994}$ are the losses in those years, then we are asked to find $r$ satisfying $(1+r)(X_{1993} \wedge 1) = (X_{1994} \wedge 1)$. By taking mathematical expectation, we have $(1+r)E(X_{1993} \wedge 1) = E(X_{1994} \wedge 1)$. So,
\[ r = \frac{E[X_{1994} \wedge 1]}{E[X_{1993} \wedge 1]} - 1 = 0.02 \quad \Rightarrow \quad r = 2\% \]

**Example**. The unlimited severity distribution for claim amounts under an auto liability insurance policy is given by the cumulative distribution:

\[
F(x) = 1 - 0.8e^{-0.02x} - 0.2e^{-0.001x}, \quad X \geq 0
\]

The insurance policy pays amounts up to a limit of 1000 per claim. Calculate the expected payment under this policy for one claim.

**Solution**.

\[
S(x) = 0.8e^{-0.02x} + 0.2e^{-0.001x}
\]

\[
E(X \wedge 1000) = \int_0^{1000} (0.8e^{-0.02x} + 0.2e^{-0.001x}) \, dx
\]

\[
= 0.8 \int_0^{1000} e^{-0.02x} \, dx + 0.2 \int_0^{1000} e^{-0.001x} \, dx
\]

Now you can calculate this integral directly, or note that the first integral is just \( E(\text{Exponential}(50) \wedge 1000) \) and the second integral is \( E(\text{Exponential}(1000) \wedge 1000) \).

Then we use the table:

\[
E(\text{Exponential}(\theta) \wedge u) = \theta \left( 1 - e^{-\frac{u}{\theta}} \right)
\]

\[
E(\text{Exponential}(50) \wedge 1000) = 50 \left( 1 - e^{-\frac{1000}{50}} \right) = 50
\]

\[
E(\text{Exponential}(1000) \wedge 1000) = 1000 \left( 1 - e^{-\frac{1000}{1000}} \right) = 632.12
\]

Answer = \((0.8)(50) + (0.2)(632.12) = 166.42\)
Coinsurance, deductibles, and limits combined

Consider a policy that pays according to the following:

\[
\begin{align*}
0 & \quad X \leq 200 \\
X - 200 & \quad 200 < X \leq 1000 \\
800 = 1000 - 200 & \quad 1000 < X
\end{align*}
\]

The choice of 800 is to have the graph of this function continuous.

The value 1000 is called the **maximum covered loss**. This function is nothing but \(X \wedge 1000 - X \wedge 200\). The policy limit is \(800 = 1000 - 200\)

In general, we may consider a deductible \(d\) and a maximum covered loss \(u\). Then the claim amount is
This is per-loss variable. The associated per-payment variable is

\[
Y^P = \begin{cases}
  \text{undefined} & X \leq d \\
  X - d & d < X \leq u \\
  u - d & \text{policy limit} \quad u < X
\end{cases}
\]

**Theorem.** For the per-loss variable defined above we have

\[
E[(Y^L)^2] = E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d E[X \wedge u] + 2d E[X \wedge d]
\]

For the second moment of per-payment variable this value must be divided by \(S(d)\)

\[
E[(Y^P)^2] = \frac{E[(Y^L)^2]}{S(d)}
\]

**Proof.**

\[
(Y^L)^2 = (X \wedge u - X \wedge d)^2 = (X \wedge u)^2 + (X \wedge d)^2 - 2(X \wedge u)(X \wedge d)
\]

\[
= (X \wedge u)^2 - (X \wedge d)^2 + 2(X \wedge d)^2 - 2(X \wedge u)(X \wedge d)
\]

\[
= (X \wedge u)^2 - (X \wedge d)^2 - 2(X \wedge d) \left[ (X \wedge u) - (X \wedge d) \right] \quad (*)
\]

But one can check that for all possibilities

\[
\begin{cases}
  0 \leq X \leq d \\
  d < X \leq u \\
  u < X
\end{cases}
\]

we have
\[2(X \wedge d) \left[ (X \wedge u) - (X \wedge d) \right] = 2d [X \wedge u - X \wedge d]\]

Then by substituting into (*) the right-hand side of (*) simplifies to

\[= (X \wedge u)^2 - (X \wedge d)^2 - 2d [X \wedge u - X \wedge d]\]

which completes the proof by taking expectation.

**Example**. Determine the mean and standard deviation per loss for a Pareto distribution with \(\alpha = 3\) and \(\theta = 2,000\) with a deductible of 500 and a policy limit of 2,500. Note that the maximum covered loss is \(u = 3,000\).

**Solution.** We recall that for \(\alpha \neq 1\) we have

\[E(X \wedge x) = \frac{\theta}{\alpha - 1} \left[ 1 - \left( \frac{\theta}{x + \theta} \right)^{\alpha-1} \right]\]

Then for \(x = 500\) and \(x = 3000\) we get \(E(X \wedge 500) = 360\) and \(E(X \wedge 3000) = 840\).

Further:

\[f(x) = \frac{\alpha \theta^\alpha}{(x + \theta)^{\alpha+1}} = \frac{3(2000)^3}{(x + 2000)^4}\]

\[\Rightarrow E[(X \wedge u)^2] = \int_0^u x^2 \frac{3(2000)^3}{(x + 2000)^4} \, dx + u^2 \left( \frac{2000}{u + 2000} \right)^3\]

\[= 3(2000)^3 \int_{2000}^{u+2000} (y - 2000)^2 y^{-4} \, dy + u^2 \left( \frac{2000}{u + 2000} \right)^3\]

\[= 3(2000)^3 \int_{2000}^{u+2000} \left( y^2 - 2(2000)y + (2000)^2 \right) y^{-4} \, dy + u^2 \left( \frac{2000}{u + 2000} \right)^3\]

\[= 3(2000)^3 \int_{2000}^{u+2000} \left( y^{-2} - 2(2000)y^{-3} + (2000)^2 y^{-4} \right) + u^2 \left( \frac{2000}{u + 2000} \right)^3\]
\[
3(2000)^3 \left[-y^{-1} + 2000y^{-2} - \frac{(2000)^2}{3}y^{-3}\right]_{y=2000}^{y=u+2000}
\]

\[
\]

Substituting \( u = 500 \) and \( u = 3000 \) gives us \( E[(X \wedge 500)^2] = 160,000 \) and \( E[(X \wedge 3000)^2] = 1,440,000 \)

\[
E[(Y^L)^2] = E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2dE[X \wedge u] + 2dE[X \wedge d]
\]

\[
= 1,440,000 - 160,000 - 2(500)(840) + 2(500)(360) = 800,000
\]

\[
\text{Var}(Y^L) = 800,000 - (480)^2 = 569,600 \quad \checkmark
\]

If there is inflation involved, then the per-loss variable will be

\[
Y^L = \begin{cases} 
0 & X \leq \frac{d}{1+r} \\
(1 + r)X - d & \frac{d}{1+r} < X \leq \frac{u}{1+r} \\
u - d & \text{policy limit} \quad \frac{u}{1+r} < X
\end{cases}
\]

Here we want to combine this with a coinsurance. In a policy with a **coinsurance factor** \( 0 < \alpha < 1 \) the insurer pays the portion \( \alpha X \) of the loss. In the presence of a deductible, a policy limit, and coinsurance, the coinsurance factor must be applied last. Under this convention, the associated per-loss variable is:

\[
Y^L = \begin{cases} 
0 & X \leq \frac{d}{1+r} \\
\alpha[(1 + r)X - d] & \frac{d}{1+r} < X \leq \frac{u}{1+r} \\
\alpha(u - d) & \text{policy limit} \quad \frac{u}{1+r} < X
\end{cases}
\]
\[
\alpha(1 + r) \begin{cases} 
0 & X \leq \frac{d}{1+r} \\
\left( X - \frac{d}{1+r} \right) & \frac{d}{1+r} < X \leq \frac{u}{1+r} \\
\left( \frac{u}{1+r} - \frac{d}{1+r} \right) & \frac{u}{1+r} < X
\end{cases}
\]

\[\text{policy limit}\]

\[
\alpha(1 + r) \left[ \left( X \wedge \frac{u}{1+r} \right) - \left( X \wedge \frac{d}{1+r} \right) \right]
\]

and the per-payment variable is:

\[
Y^L = \begin{cases} 
\text{undefined} & X \leq \frac{d}{1+r} \\
\alpha(1 + r)X - d & \frac{d}{1+r} < X \leq \frac{u}{1+r} \\
\alpha(u - d) & \frac{u}{1+r} < X
\end{cases}
\]

\[
\text{policy limit}
\]

Now the proof of the following theorem should be clear:

**Theorem.** For the variables such defined, we have

\[
E(Y^L) = \alpha(1 + r) \left[ E\left( X \wedge \frac{u}{1+r} \right) - E\left( X \wedge \frac{d}{1+r} \right) \right]
\]

\[
E(Y^P) = \frac{E(Y^L)}{S_X\left( \frac{d}{1+r} \right)}
\]

The proof of the following theorem is similar to the proof of a similar theorem proved above.

**Theorem.** For the variables such defined, we have

\[
E\left[ (Y^L)^2 \right] = \alpha(1 + r)^2 \left\{ E\left[ (X \wedge u^*)^2 \right] - E\left[ (X \wedge d^*)^2 \right] - 2d^* E\left[ X \wedge u^* \right] + 2d^* E\left[ X \wedge d^* \right] \right\}
\]

\[
E\left[ (Y^P)^2 \right] = \frac{E\left[ (Y^L)^2 \right]}{S(d^*)}
\]

where \( u^* = \frac{u}{1+r} \) and \( d^* = \frac{d}{1+r} \).
**Note.** Examples 35.2, 35.3, 35.5, and 35.8 of the Finan’s study guide have been solved in class or you are expected to know them if not done in class.