Absolute Maxima and Minima
sections 4.7

Definition. Let us study a function f on an interval I.

(i) We say that the function f has **absolute maximum** at a point $x_0 \in I$ if for all $x \in I$ we have $f(x) \leq f(x_0)$.

(ii) We say that the function f has **absolute minimum** at a point $x_0 \in I$ if for all $x \in I$ we have $f(x) \geq f(x_0)$.

Question. Under what circumstances it is guaranteed that a function has absolute extrema on an interval I ? The following is an answer to this question.

Theorem. If f is continuous on a closed interval $[a, b]$, then it has an absolute maximum and an absolute minimum on $[a, b]$.

Question. Now that we know the circumstances , how could we actually search for the points of absolute maxima?. The following theorem answers this question:

Theorem. The $f(x_0)$ is an absolute extrema of function on an interval I, then x_0 is amongst one of the following points:

- x_0 is an endpoint of the interval I .
- x_0 is an interior point at which the derivative does not exist.
- x_0 is an interior point at which we have $f'(x_0) = 0$.

Note. In the above theorem , the points which fall into the second and third categories are called **critical points**. Geometrically , these are the points at which either a tangent line does not exist or the tangent line is horizontal.

Note. The Figure 4.17 on page 249 was explained in class.

Example (section 4.7 exercise 3). Find the absolute extrema of the function $f(x) = x + \frac{1}{x}$ over the interval $\frac{1}{2} \leq x \leq 5$ if they exist.

Solution.

Step 1. Because the function is continuous over the closed interval $[\frac{1}{2}\,,\,5]$, according the above theorem the absolute extrema exist. Now we find them:

Step 2.

$f(x) = x + x^{-1}$

$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$

Step 3.

$f'(x) = 0 \quad \Rightarrow \quad 1 - \frac{1}{x^2} = 0 \quad \Rightarrow \quad \frac{1}{x^2} = 1 \quad \Rightarrow \quad x^2 = 1$

$$\Rightarrow \quad \begin{cases} x = 1 \\ x = -1 \quad \textbf{this is not acceptable ; not in the domain} \end{cases}$$

According to the theorem , one of the candidates is $x = 1$ and two others are the endpoints $x = \frac{1}{2}$ and $x = 5$. Other candidates are the points of the interval at which the derivative does not exist , but the function is non-differentiable at the origin only and the origin is not in the domain. So the only candidates are $x = 1$ and $x = \frac{1}{2}$ and $x = 5$.

Step 4. Now the table:

candidate x	$f(x)$	
$\frac{1}{2}$	2.5	
1	2	**absolute min**
5	5.2	**absolute max**

Example (section 4.7 exercise 5). Find the absolute extrema of the function $f(x) = x\sqrt{x+1}$ over the interval $-1 \le x \le 1$ if they exist.

Solution.

Step 1. Because the function is continuous over the closed interval $[-1\,,\,1]$, according the above theorem the absolute extrema exist. Now we find them:

Step 2.

$$f(x) = x(x+1)^{\frac{1}{2}}$$

$$\begin{aligned}
f'(x) &= \Big\{x\Big\}'\Big\{(x+1)^{\frac{1}{2}}\Big\} + \Big\{x\Big\}\Big\{(x+1)^{\frac{1}{2}}\Big\}' \\
&= \Big\{1\Big\}\Big\{(x+1)^{\frac{1}{2}}\Big\} + \Big\{x\Big\}\Big\{\tfrac{1}{2}(x+1)^{-\frac{1}{2}}\Big\} \\
&= \sqrt{x+1} + \tfrac{x}{2\sqrt{x+1}} \\
&= \tfrac{2\left(\sqrt{x+1}\right)^2 + x}{2\sqrt{x+1}} \\
&= \tfrac{2(x+1)+x}{2\sqrt{x+1}} \\
&= \tfrac{3x+2}{2\sqrt{x+1}}
\end{aligned}$$

Step 3.

$f'(x) = 0 \quad \Rightarrow \quad 3x + 2 \quad \Rightarrow \quad x = -\frac{2}{3}$

According to the theorem , one of the candidates is $x = -\frac{2}{3}$ and two others are the endpoints $x = -1$ and $x = 1$. Other candidates are the points of the interval at which the derivative does not exist , but the derivative $f'(x) = \frac{3x+2}{2\sqrt{x+1}}$ does not exist at the point $x = -1$ only , which is already one of our candidates , so we do not get any new candidates. So finally the only candidates are $x = -\frac{2}{3}$ and $x = -1$ and $x = 1$.

Step 4. Now the table:

candidate x	$f(x)$	
-1	0	
$-\frac{2}{3}$	$\frac{-2}{3\sqrt{3}}$	**absolute min**
1	$\sqrt{2}$	**absolute max**

What if the function is continuous but the interval in not closed?

The next examples are of this type and will show how to tackle these problems.

Example. Find the absolute extrema of the function $f(x) = \frac{1}{x^2+1}$ over the interval $[1\,,\,2)$ if they exist.

Solution.

Step 1.

$f'(x) = \frac{(1)'(x^2+1)-(1)(x^2+1)'}{(x^2+1)^2} = \frac{-2x}{(x^2+1)^2}$

$f'(x) = 0 \quad \Rightarrow \quad x = 0$ **not acceptable ; it is not in the domain**

Step 2.

There are no non-differentiability candidates.

Step 3.

Since the point $x = 2$ is an endpoint but not in the domain , then we calculate the limit at this point:

$$\lim_{x\to 2^-} \frac{1}{x^2+1} = \frac{1}{5}$$

Step 4.

The endpoint $x = 1$ is a candidate. So we have only one candidate $x = 1$. We must check whether this candidate can be either an absolute min or an absolute max point:

	candidate x	$f(x)$ or $\lim f(x)$	
	1	$\frac{1}{2}$	**absolute max**
(not a candidate)	2	$\frac{1}{5}$	

Example. Repeat the above question , but this time over the interval $(1\ ,\ 2)$; i.e., find the absolute extrema of the function $f(x) = \frac{1}{x^2+1}$ over the interval $(1\,,\,2)$ if they exist.

Solution.

Most of the details are as above except that the point $x = 1$ is not a candidate because is not in the domain. So there is no candidate in this case , so there is no absolute max or absolute min for this function over the interval $(1\,,\,2)$

Example. Repeat the above question , but this time over the interval $[1\,,\,2]$; i.e., find the absolute extrema of the function $f(x) = \frac{1}{x^2+1}$ over the interval $[1\,,\,2]$ if they exist.

Solution.

This time , the points are $x = 1$ and $x = 2$ are the only candidates.

candidate x	$f(x)$	
1	$\frac{1}{2}$	**absolute max**
2	$\frac{1}{5}$	**absolute min**

Example. Find the absolute extrema of the function $f(x) = x^3 - \frac{9}{2}x^2 + 6x + 1$ over the interval $[\frac{3}{2}\,,\,\infty)$ if they exist.

Solution.

First Step. Check for possible points in the domain that make f' zero:

$f'(x) = 3x^2 - 9x + 6 = 3(x^2 - 3x + 2)$

$$f'(x) = 0 \quad \Rightarrow \quad x = \tfrac{3 \pm \sqrt{9-8}}{2} = \tfrac{3 \pm 1}{2} = \begin{cases} 1 & \textbf{not acceptable} \\ 2 \end{cases}$$

Next Step. Check for possible points in the domain at which f' does not exist:

There are none.

Next Step. Deal with the end points :

$$\begin{cases} \lim\limits_{x \to \infty} x^3 - \frac{9}{2}x^2 + 6x + 1 = \infty \\ \text{a possible candidate is} x = \frac{3}{2} \end{cases}$$

Next Step. Form the table

	candidate x	$f(x)$ or $\lim f(x)$	
	$\frac{3}{2}$	$\frac{53}{8}$	
	2	3	**absolute min**
(not a candidate)	∞	∞	

<u>**Example**</u>. Find the absolute extrema of the function $f(x) = \sqrt[3]{x} - \frac{1}{3}x$ over

the interval $(-8\,,\,8]$ if they exist.

Solution.

$f(x) = x^{\frac{1}{3}} - \frac{1}{3}x$

First Step. Check for possible points in the domain that make f' zero:

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{3} = \frac{1}{3\sqrt[3]{x^2}} - \frac{1}{3}$

$f'(x) = 0 \quad \Rightarrow \quad \frac{1}{3\sqrt[3]{x^2}} - \frac{1}{3} \quad \Rightarrow \quad \frac{1}{3\sqrt[3]{x^2}} = \frac{1}{3} \quad \Rightarrow \quad \sqrt[3]{x^2} = 1$

$\Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1\ .$

Next Step. Check for possible points in the domain at which f' does not exist:

x=0

Next Step. Deal with the end points :

$$\begin{cases} \lim\limits_{x \to (-8)^+} f(x) = -2 + \frac{8}{3} = \frac{2}{3} \\ f(8) = -\frac{2}{3} \end{cases}$$

Next Step. Form the table

	candidate x	$f(x)$ or $\lim f(x)$	
(not a candidate)	-8	$\frac{2}{3}$	
	-1	$-\frac{4}{3}$	**absolute min**
	0	0	
	1	$\frac{2}{3}$	**absolute max**
	8	$-\frac{2}{3}$	