Reverse Operation of Differentiation
section 5.1

Definition. Suppose that function $f(x)$ is defined on an interval I. A function $F(x)$ is said to be an **antiderivative** of f if

$$F'(x) = f(x) \qquad \text{for all } x \text{ in } I$$

Note. The antiderivative is not unique. For example both functions $\frac{1}{3}x^3$ and $\frac{1}{3}x^3 + 1$ are antiderivatives of the function x^2 over the interval $(-\infty\,,\,\infty)$. In fact, all functions $\frac{1}{3}x^3+C$, where C is an arbitrary constant, is an antiderivative of x^2. The next theorem shows that any antiderivative is found by adding a constant to an antiderivative.

Theorem. Any two antiderivative of f over interval I differ by a constant, i.e. , if $F(x)$ is an antiderivative , then every antiderivative is of the form $F(x) + C$ where C is a constant (not dependent on x).

Proof. Let $G(x)$ be any antiderivative of $f(x)$ over the interval I, We want to show that there is some C such that $G(x) = F(x) + C$ for all $x \in I$. To show this , take the function

$$H(x) = G(x) - F(x)$$

By taking the derivative of both sides of this equality , we will have

$$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0 \qquad \text{for all } x \in I \qquad (1)$$

So the derivative of H is zero over the interval I.

Fix a point $a \in I$, and take the arbitrary point $x \in I$ different from a. According to the **Mean-Value Theorem** there exists some c between a and x such that

$$\frac{H(x) - H(a)}{x - a} = H'(c) \qquad (2)$$

But from (1) we have $H'(c) = 0$, therefore the equality (2) reduces to $\frac{H(x)-H(a)}{x-a} = 0$, equivalently $H(x) - H(a) = 0$, and then $H(x) = H(a)$. But then we have

$$H(x) = H(a) \;\Rightarrow\quad G(x) - F(x) = G(a) - F(a)$$

$$\Rightarrow\quad G(x) = F(x) + \underbrace{G(a) - F(a)}_{C}$$

So , we have provided a constant C , namely $C = G(a) - F(a)$, not dependent on x , such that for all $x \in I$ we have $G(x) = F(x) + C$. This is exactly what we were asked to prove. ✓

Definition. The collection of all antiderivatives of a given function f is denoted by $\int f(x)\,dx$ and is called the **indefinite integral** of $f(x)$.

Example. Since the derivative of $\tan x$ is $\sec^2 x$, equivalently , an antiderivative of the function $\sec^2 x$ is $\tan x$, and then all the antiderivatives can be found by adding a constant:

$$\int \sec^2 x \, dx = \tan x + C$$

we can write:

$$\frac{d(\tan x)}{dx} = \sec^2 x \quad \Leftrightarrow \quad \int \sec^2 x \, dx = \tan x + C$$

As another example , for any real number $r \neq -1$ we have

$$\frac{d}{dx}\left(\frac{1}{r+1}\, x^{r+1}\right) = x^r \quad \Rightarrow \quad \int x^r \, dx = \frac{1}{r+1}\, x^{r+1} + C$$

Example.

$$\int x^{\sqrt{3}} dx = \frac{1}{\sqrt{3}+1}\, x^{\sqrt{3}+1} + C$$

Algebraic Properties of Indefinite Integral.

$$\int \Big[f(x) + g(x)\Big] \, dx = \int f(x) \, dx + \int g(x) \, dx$$

$$\int k \, f(x) \, dx = k \int f(x) \, dx$$

Example (section 5.1 exercise 8).

Find $\displaystyle\int \left(\frac{1}{x^2} - \frac{2}{x^4}\right) dx$

Solution.

$$= \int \left(x^{-2} - 2x^{-4}\right) dx$$

$$= \int x^{-2}\, dx - 2 \int x^{-4} dx$$

$$= \frac{1}{-1}\, x^{-1} - 2\left(\frac{1}{-3}\, x^{-3}\right) + C$$

$$= -\frac{1}{x} + \frac{2}{3x^3} + C$$

Example (section 5.1 exercise 20). Evaluate $\displaystyle\int \frac{(x-1)^2}{\sqrt{x}}\, dx$

Solution.

$$= \int \frac{x^2 - 2x + 1}{\sqrt{x}}\, dx$$

$$= \int \left(\frac{x^2}{\sqrt{x}} - 2\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx$$

$$= \int \left(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx$$

$$= \frac{2}{5}x^{\frac{5}{2}} - 2(\frac{2}{3})x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

$$= \frac{2}{5}\, x^{\frac{5}{2}} - \frac{4}{3}\, x^{\frac{3}{2}} + 2\, x^{\frac{1}{2}} + C$$

Example (section 5.1 exercise 25).

Find the equation of the curve that has a second derivative equal to $6x^2$ and passes through the points $(0,2)$ and $(-1,3)$

Solution.

$y'' = 6x^2$

Integrate:

$y' = 6\left(\frac{1}{3}x^3\right) + C = 2x^3 + C$

Integrate again:

$y = 2\left(\frac{1}{4}x^4\right) + Cx + D = \frac{1}{2}x^4 + Cx + D$

Insert the point $(0,2)$ to have:

$2 = D$

Substitute $D = 2$ in $y = \frac{1}{2}x^4 + Cx + D$ to have:

$y = \frac{1}{2}x^4 + Cx + 2$

Insert the point $(-1,3)$ to have:

$3 = \frac{1}{2} - C + 2 \quad \Rightarrow \quad C = -\frac{1}{2}$

Substitute $C = -\frac{1}{2}$ in $y = \frac{1}{2}x^4 + Cx + 2$ to have:

$y = \frac{1}{2}x^4 - \frac{1}{2}x + 2 \qquad \checkmark$

Example (section 5.1 exercise 27).

Is it possible to find a function that has a relative minimum $f(2) = 3$ and has a second derivative equal to $-5x$?

Solution.

$f''(x) = -5x \quad \Rightarrow \quad f''(2) = -10 < 0$

$\Rightarrow$ (**second derivative test**) the point $x = 2$ gives a local maximum

$\Rightarrow$ no such function exists.

A table of indefinite integral you are supposed to know for your your final exam:

$$\left\{ \begin{aligned}
\int x^r \, dx &= \tfrac{1}{r+1}\, x^{r+1} + C \qquad r \neq -1 \\
\int \frac{1}{x} \, dx &= \ln|x| + C \\
\int e^x \, dx &= e^x + C \\
\int a^x \, dx &= \tfrac{1}{\ln(a)}\, a^x + C \\
\int \sin x \, dx &= -\cos x + C \\
\int \cos x \, dx &= \sin x + C \\
\int \sec^2 x \, dx &= \tan x + C \\
\int \sec x \; \tan x \, dx &= \sec x + C \\
\int \csc^2 x \, dx &= -\cot x + C \\
\int \csc x \; \cot x \, dx &= -\csc x + C
\end{aligned} \right.$$