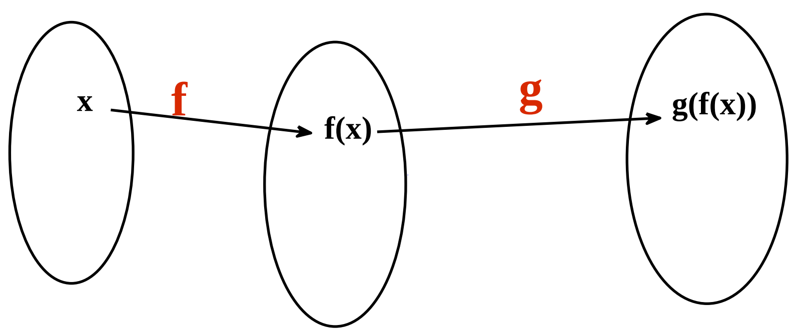
The Chain Rule and the Extended Power Rule
section 3.7

Theorem (Chain Rule)): Suppose that the function f is differentiable at a point x and that g is differentiable at $f(x)$. Then the function $g \circ f$ is differentiable at x and we have

$$(g \circ f)'(x) = g'(f(x))\, f'(x)$$



Note: So, if the derivatives on the right-hand side of the above equality exist , then the derivative on the left-hand side exists and the above equality holds. If we put $u = f(x)$ and $y = (g)(u)$, then we have

$$y = g(u) = g(f(x)) = (g \circ f)(x)$$

So by the convention , y is just the function $(g \circ f)(x)$, and therefore by the chain rule its derivative equals

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\begin{tikzcd} y \arrow[d, dash] \\ u \arrow[d, dash] \\ x \end{tikzcd}$$

Theorem (power rule for all rational exponents): If $\frac{m}{n}$ is any rational number (where m is an arbitrary integer , and n is an arbitrary positive integer) , then

$$\left\{x^{\frac{m}{n}}\right\}' = \frac{m}{n} x^{\frac{m}{n}-1}$$

Proof: Set $y = x^{\frac{m}{n}}$. We want to prove that $\frac{dy}{dx} = \frac{m}{n} x^{\frac{m}{n}-1}$. We use the chain rule to prove this. In fact , note that

$$y = \left(x^{\frac{1}{n}}\right)^m = u^m \qquad \text{where } u = x^{\frac{1}{n}}$$

Then

$$\begin{aligned}
\tfrac{dy}{dx} &= \tfrac{dy}{du}\,\tfrac{du}{dx} \\
&= (mu^{m-1})\tfrac{du}{dx} && \text{power rule for positive integers} \\
&= (mu^{m-1})\left(\tfrac{1}{n}x^{\frac{1}{n}-1}\right) && \text{power rule for the exponents of the form } \tfrac{1}{n} \\
&= (mu^{m-1})\left(\tfrac{1}{n}x^{\frac{1-n}{n}}\right) \\
&= \left(m\left\{x^{\frac{1}{n}}\right\}^{m-1}\right)\left(\tfrac{1}{n}x^{\frac{1-n}{n}}\right) \\
&= \left(m\,x^{\frac{m-1}{n}}\right)\left(\tfrac{1}{n}x^{\frac{1-n}{n}}\right) \\
&= \tfrac{m}{n}\,x^{\frac{m-1}{n}+\frac{1-n}{n}} \\
&= \tfrac{m}{n}\,x^{\frac{m}{n}-1} && \checkmark
\end{aligned}$$

Theorem (power rule for all real exponents): If a is any real number, then $\frac{d}{dx}x^a = ax^{a-1}$.

Note: This fact will be studied in section 3.11 extensively. As some examples we have

$$f(x) = x^{-\frac{2}{3}} \qquad \Rightarrow \qquad f'(x) = -\frac{2}{3}x^{-\frac{5}{3}}$$

$$f(x) = x^{\sqrt{2}} \qquad \Rightarrow \qquad f'(x) = \sqrt{2}\; x^{\sqrt{2}-1}$$

Example: Find the derivative of the function $f(t) = \left(\frac{2t+1}{t-1}\right)^7$

Solution: **Convention**: Here in this solution, the prime notation refers to the derivative with respect to the variable t. Write

$$f(t) = u^7 \qquad \text{where} \qquad u = \frac{2t+1}{t-1}$$

Then

$$\begin{aligned}
f'(t) &= \tfrac{d(u^7)}{du}\,\tfrac{du}{dt} && \text{chain rule}\\
&= (7u^6)\,\tfrac{du}{dt} && \text{power rule}\\
&= (7u^6)\,\tfrac{\{2t+1\}'\{t-1\}-\{2t+1\}\{t-1\}'}{(t-1)^2}\\
&= (7u^6)\,\tfrac{\{2\}\{t-1\}-\{2t+1\}\{1\}}{(t-1)^2}\\
&= (7u^6)\left(\tfrac{-3}{(t-1)^2}\right)\\
&= 7\left(\tfrac{2t+1}{t-1}\right)^6\left(\tfrac{-3}{(t-1)^2}\right) && \text{change to the original parameter } t\\
&= \tfrac{-21(2t+1)^6}{(t-1)^8} \quad \checkmark
\end{aligned}$$

Extended power rule: If a is any real number (rational or irrational), then

$$\frac{d}{dx}g(x)^a = ag(x)^{a-1}\,g'(x)$$

derivative of $\mathbf{g(x)^a}$ = (the simple power rule) $\times$ (derivative of the function inside)

Note: This theorem has appeared on page 189 of the textbook. The proof of it is easy as one can take $u = g(x)$ and then apply the chain rule. This theorem is very handy. See the next example :

Example (from the textbook): Differentiate the function $y = (2x^2 - 3)^8$.

Solution: **Convention**: Here in this solution, the prime notation refers to the derivative with respect to the variable x. Then

$$y' = \{8(2x^2-3)^7\}\,\{2x^2-3\}' = \{8(2x^2-3)^7\}\,\{4x\} = 32x(2x^2-3)^7 \qquad \checkmark$$

<u>**Example (section 3.7 exercise 34)**</u>: Find $\frac{dy}{dx}$ if

$$y = t + \sqrt{t + \sqrt{t}} \quad \text{where} \quad t = \frac{x^2+1}{x^2-1}$$

<u>**Solution**</u>: <u>**Step 1**</u>: <u>**Convention**</u>: Here in this solution, the prime notation refers to the derivative with respect to the variable x.

Then

$$\frac{dt}{dx} = \frac{\{x^2+1\}'\{x^2-1\} - \{x^2+1\}\{x^2-1\}'}{(x^2-1)^2} = \frac{\{2x\}\{x^2-1\} - \{x^2+1\}\{2x\}}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

<u>**Step 2**</u>: Note that $y = t + (t + t^{\frac{1}{2}})^{\frac{1}{2}}$. Then

$$\begin{aligned}
y' &= t' + \left\{(t + t^{\frac{1}{2}})^{\frac{1}{2}}\right\}' = \tfrac{-4x}{(x^2-1)^2} + \left\{\tfrac{1}{2}(t + t^{\frac{1}{2}})^{-\frac{1}{2}}\right\}\left\{t + t^{\frac{1}{2}}\right\}' \\
&= \tfrac{-4x}{(x^2-1)^2} + \left\{\tfrac{1}{2}(t + t^{\frac{1}{2}})^{-\frac{1}{2}}\right\}\left\{t' + \tfrac{1}{2}t^{-\frac{1}{2}}\right\} \\
&= \tfrac{-4x}{(x^2-1)^2} + \left\{\tfrac{1}{2}(t + t^{\frac{1}{2}})^{-\frac{1}{2}}\right\}\left\{\tfrac{-4x}{(x^2-1)^2} + \tfrac{1}{2}t^{-\frac{1}{2}}\right\} \quad \text{where} \quad t = \tfrac{x^2+1}{x^2-1}
\end{aligned}$$

<u>Note</u>: In the last equality of the above example we wrote

$$\text{where} \quad t = \frac{x^2+1}{x^2-1}$$

because we cannot simplify the last expression further.