The Chain Rule and the Extended Power Rule
section 3.7

**Theorem (Chain Rule):** Suppose that the function \( f \) is differentiable at a point \( x \) and that \( g \) is differentiable at \( f(x) \). Then the function \( g \circ f \) is differentiable at \( x \) and we have

\[
(g \circ f)'(x) = g'(f(x)) f'(x)
\]

**Note:** So, if the derivatives on the right-hand side of the above equality exist, then the derivative on the left-hand side exists and the above equality holds. If we put \( u = f(x) \) and \( y = (g)(u) \), then we have

\[
y = g(u) = g(f(x)) = (g \circ f)(x)
\]

So by the convention, \( y \) is just the function \( (g \circ f)(x) \), and therefore by the chain rule its derivative equals

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
\]
Theorem (power rule for all rational exponents): If \( \frac{m}{n} \) is any rational number (where \( m \) is an arbitrary integer, and \( n \) is an arbitrary positive integer), then

\[
\left\{ x^{\frac{m}{n}} \right\}' = \frac{m}{n} x^{\frac{m}{n} - 1}
\]

Proof: Set \( y = x^{\frac{m}{n}} \). We want to prove that \( \frac{dy}{dx} = \frac{m}{n} x^{\frac{m}{n} - 1} \). We use the chain rule to prove this. In fact, note that

\[
y = \left( x^{\frac{1}{n}} \right)^m = u^m \quad \text{where} \quad u = x^{\frac{1}{n}}
\]
Then
\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
\]
\[
= (mu^{m-1}) \frac{du}{dx} \quad \text{power rule for positive integers}
\]
\[
= (mu^{m-1}) \left( \frac{1}{n} x^{\frac{1}{n} - 1} \right) \quad \text{power rule for the exponents of the form } \frac{1}{n}
\]
\[
= (mu^{m-1}) \left( \frac{1}{n} x^{\frac{1-n}{n}} \right)
\]
\[
= \left( m \left( x^{\frac{1}{n}} \right)^{m-1} \right) \left( \frac{1}{n} x^{\frac{1-n}{n}} \right)
\]
\[
= \left( mx^{\frac{m-1}{n}} \right) \left( \frac{1}{n} x^{\frac{1-n}{n}} \right)
\]
\[
= \frac{m}{n} x^{\frac{m-1}{n} + \frac{1-n}{n}}
\]
\[
= \frac{m}{n} x^{\frac{m}{n} - 1} \quad \checkmark
\]

**Theorem (power rule for all real exponents):** If \( a \) is any real number, then \( \frac{d}{dx} x^a = ax^{a-1} \).

**Note:** This fact will be studied in section 3.11 extensively. As some examples we have

\[
f(x) = x^{-\frac{2}{3}} \quad \Rightarrow \quad f'(x) = -\frac{2}{3} x^{-\frac{5}{3}}
\]

\[
f(x) = x^{\sqrt{2}} \quad \Rightarrow \quad f'(x) = \sqrt{2} x^{\sqrt{2}-1}
\]

**Example:** Find the derivative of the function \( f(t) = \left( \frac{2t+1}{t-1} \right)^7 \)

**Solution:** **Convention:** Here in this solution, the prime notation refers to the derivative with respect to the variable \( t \). Write

\[
f(t) = u^7 \quad \text{where} \quad u = \frac{2t + 1}{t - 1}
\]
Then

\[ f'(t) = \frac{d(u^7)}{du} \frac{du}{dt} \]

chain rule

\[ = (7u^6) \frac{du}{dt} \]

power rule

\[ = (7u^6) \frac{(2t+1)'(t-1)-(2t+1)(t-1)'}{(t-1)^2} \]

\[ = (7u^6) \frac{(2)(t-1)-(2t+1)(1)}{(t-1)^2} \]

\[ = (7u^6) \left( \frac{-3}{(t-1)^2} \right) \]

\[ = 7 \left( \frac{2t+1}{t-1} \right)^6 \left( \frac{-3}{(t-1)^2} \right) \]

change to the original parameter \( t \)

\[ = \frac{-21(2t+1)^6}{(t-1)^8} \]

\[ \checkmark \]

**Extended power rule:** If \( a \) is any real number (rational or irrational), then

\[ \frac{d}{dx} g(x)^a = ag(x)^{a-1} g'(x) \]

derivative of \( g(x)^a = (\text{the simple power rule}) \times (\text{derivative of the function inside}) \)

**Note:** This theorem has appeared on page 189 of the textbook. The proof of it is easy as one can take \( u = g(x) \) and then apply the chain rule. This theorem is very handy. See the next example:

**Example (from the textbook):** Differentiate the function \( y = (2x^2 - 3)^8 \).

**Solution:** **Convention:** Here in this solution, the prime notation refers to the derivative with respect to the variable \( x \). Then

\[ y' = \{8(2x^2 - 3)^7\} \{2x^2 - 3\}' = \{8(2x^2 - 3)^7\} \{4x\} = 32x(2x^2 - 3)^7 \]

\[ \checkmark \]
Example (section 3.7 exercise 34): Find $\frac{dy}{dx}$ if
\[ y = t + \sqrt{t + \sqrt{t}} \quad \text{where} \quad t = \frac{x^2 + 1}{x^2 - 1} \]

Solution: Step 1: Convention: Here in this solution, the prime notation refers to the derivative with respect to the variable $x$.

Then
\[ \frac{dt}{dx} = \frac{(x^2 + 1)'(x^2 - 1) - (x^2 + 1)(x^2 - 1)'}{(x^2 - 1)^2} = \frac{2x(x^2 - 1) - (x^2 + 1)(2x)}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2} \]

Step 2: Note that $y = t + (t + t^\frac{1}{2})^\frac{1}{2}$. Then
\[ y' = t' + \left\{ (t + t^\frac{1}{2})^\frac{1}{2} \right\}' = \frac{-4x}{(x^2 - 1)^2} + \left\{ \frac{1}{2}(t + t^\frac{1}{2})^{-\frac{1}{2}} \right\} \left\{ t + t^\frac{1}{2} \right\}' \]
\[ = \frac{-4x}{(x^2 - 1)^2} + \left\{ \frac{1}{2}(t + t^\frac{1}{2})^{-\frac{1}{2}} \right\} \left\{ t' + \frac{1}{2}t^{-\frac{1}{2}} \right\} \]
\[ = \frac{-4x}{(x^2 - 1)^2} + \left\{ \frac{1}{2}(t + t^\frac{1}{2})^{-\frac{1}{2}} \right\} \left\{ \frac{-4x}{(x^2 - 1)^2} + \frac{1}{2}t^{-\frac{1}{2}} \right\} \quad \text{where} \quad t = \frac{x^2 + 1}{x^2 - 1} \]

Note: In the last equality of the above example we wrote
\[ \text{where} \quad t = \frac{x^2 + 1}{x^2 - 1} \]
because we cannot simplify the last expression further.