

Change of Variable For Definite Integral

Section 6.7

Note. Let us go back to the example $\int_{-2}^4 \sqrt{x+4} dx$ we worked on in the last lecture. There we made the change of variable $u = x + 4$. Using this change of variable, we changed the associated indefinite integral to the indefinite integral $\frac{2}{3}u^{\frac{3}{2}} + C$ and then change it to something in terms of x . Another option is this:

First: find the associated values for u for the upper and lower limits of the integral $\int_{-2}^4 \sqrt{x+4} dx$ which are

$$\left\{ \begin{array}{ll} x = 4 & \xRightarrow{u=x+4} u = 8 \\ x = -2 & \xRightarrow{u=x+4} u = 2 \end{array} \right.$$

Second: continue in this way:

$$\begin{aligned} \int_{-2}^4 \sqrt{x+4} dx &= \int_2^8 \sqrt{u} du = \int_2^8 u^{\frac{1}{2}} du = \left[\frac{2}{3} u^{\frac{3}{2}} \right]_{u=2}^{u=8} = \frac{2}{3} \left[u^{\frac{3}{2}} \right]_{u=2}^{u=8} \\ &= \frac{2}{3} [16\sqrt{2} - 2\sqrt{2}] = \frac{28\sqrt{2}}{3} \end{aligned}$$

which is the same answer as above. In this method we do not calculate the indefinite integral involving x . This is the whole story portrayed by Theorem 6.9 on page 405.

Example (section 6.7 exercise 15). Evaluate the integral

$$\int_4^9 \sqrt{1 + \sqrt{x}} \, dx$$

Solution.

$$u = 1 + \sqrt{x} = 1 + x^{\frac{1}{2}} \quad \Rightarrow \quad du = \frac{1}{2}x^{-\frac{1}{2}} \, dx = \frac{1}{2\sqrt{x}} \, dx$$

$$\Rightarrow \quad dx = 2\sqrt{x} \, du = 2(u - 1)du$$

Next Step.

$$\begin{cases} x = 9 & \Rightarrow & u = 4 \\ x = 4 & \Rightarrow & u = 3 \end{cases}$$

Next Step.

$$\begin{aligned} \int_4^9 \sqrt{1 + \sqrt{x}} \, dx &= \int_3^4 2\sqrt{u}(u - 1) \, du = 2 \int_3^4 u^{\frac{1}{2}}(u - 1) \, du \\ &= 2 \int_3^4 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}} \right) \, du = 2 \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} \right]_{u=3}^{u=4} = 2 \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} \right]_{u=3}^{u=4} \\ &= 2 \left\{ \left(\frac{64}{5} - \frac{16}{3} \right) - \left(\frac{18\sqrt{3}}{5} - 2\sqrt{3} \right) \right\} = \frac{224 - 48\sqrt{3}}{15} \end{aligned}$$

Theorem. For three points a and b and c the following equality holds provided that the three integrals exist:

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$$

Note. We will use the above theorem in the next example to evaluate the given integral:

Example (section 6.7 exercise 17). Evaluate the integral

$$\int_{-1}^1 \frac{|x|}{(x+2)^3} dx$$

Solution.

$$\int_{-1}^1 \frac{|x|}{(x+2)^3} dx = \underbrace{\int_{-1}^0 \frac{(-x)}{(x+2)^3} dx}_{\mathbf{A}} + \underbrace{\int_0^1 \frac{x}{(x+2)^3} dx}_{\mathbf{B}}$$

Next Step.

$$\mathbf{A} = - \int_{-1}^0 \frac{x}{(x+2)^3} dx \quad \begin{cases} u = x + 2 \\ du = dx \end{cases} \quad \begin{cases} x = 0 & \Rightarrow u = 2 \\ x = -1 & \Rightarrow u = 1 \end{cases}$$

$$= - \int_1^2 \frac{u-2}{u^3} du = - \int_1^2 (u-2)u^{-3} du = - \int_1^2 (u^{-2} - 2u^{-3}) du$$

$$= - \left[\frac{u^{-1}}{-1} - 2 \left(\frac{u^{-2}}{-2} \right) \right]_1^2 = - \left[-\frac{1}{u} + \frac{1}{u^2} \right]_1^2 = \left[\frac{1}{u} - \frac{1}{u^2} \right]_1^2$$

$$= \left(\frac{1}{2} - \frac{1}{4} \right) - (1 - 1) = \frac{1}{4}$$

Next Step.

$$\begin{aligned}
\mathbf{B} &= \int_0^1 \frac{x}{(x+2)^3} dx \quad \begin{cases} u = x + 2 \\ du = dx \end{cases} \quad \begin{cases} x = 1 \Rightarrow u = 3 \\ x = 0 \Rightarrow u = 2 \end{cases} \\
&= \int_2^3 \frac{u-2}{u^3} du = \int_2^3 (u-2)u^{-3} du = \int_2^3 (u^{-2} - 2u^{-3}) du \\
&= \left[\frac{u^{-1}}{-1} - 2 \left(\frac{u^{-2}}{-2} \right) \right]_2^3 = \left[-\frac{1}{u} + \frac{1}{u^2} \right]_2^3 = \left(-\frac{1}{3} + \frac{1}{9} \right) - \left(-\frac{1}{2} + \frac{1}{4} \right) = \frac{1}{36}
\end{aligned}$$

Next Step.

$$\int_{-1}^1 \frac{|x|}{(x+2)^3} dx = \mathbf{A} + \mathbf{B} = \frac{1}{4} + \frac{1}{36} = \frac{5}{18}$$

Example (section 6.7 exercise 19). Evaluate the integral $\int_0^1 x^2 e^{x^3} dx$

Solution.

$$u = x^3 \Rightarrow du = 3x^2 dx \Rightarrow x^2 dx = \frac{1}{3} du \Rightarrow x^2 e^{x^3} dx = \frac{1}{3} e^u du$$

or do this:

$$u = x^3 \Rightarrow du = 3x^2 dx \Rightarrow dx = \frac{1}{3x^2} du = \frac{1}{3(\sqrt[3]{u})^2} du$$

$$\Rightarrow x^2 e^{x^3} dx = (\sqrt[3]{u})^2 e^u \frac{1}{3(\sqrt[3]{u})^2} du = \frac{1}{3} e^u du$$

Next Step.

$$\begin{cases} x = 1 & \Rightarrow & u = 1 \\ x = 0 & \Rightarrow & u = 0 \end{cases}$$

Next Step.

$$\begin{aligned} \int_{x=0}^{x=1} x^2 e^{x^3} dx &= \int_{u=0}^{u=1} \frac{1}{3} e^u du = \frac{1}{3} \int_{u=0}^{u=1} e^u du \\ &= \frac{1}{3} \left[e^u \right]_{u=0}^{u=1} = \frac{1}{3} (e^1 - e^0) = \frac{e-1}{3} \end{aligned}$$

Example (section 6.7 exercise 8). Evaluate the integral

$$\int_{1/2}^1 \sqrt{\frac{x^2}{1+x}} dx$$

Solution.

Write the integral in the form

$$\int_{1/2}^1 \frac{x}{\sqrt{1+x}} dx$$

and then make the following change of variable:

$$\begin{cases} u = 1 + x \\ du = dx \end{cases} \quad \begin{cases} x = 1 & \Rightarrow & u = 2 \\ x = \frac{1}{2} & \Rightarrow & u = \frac{3}{2} \end{cases}$$

Next Step.

$$\begin{aligned}
\int_{1/2}^1 \frac{x}{\sqrt{1+x}} dx &= \int_{3/2}^2 \frac{u-1}{\sqrt{u}} du = \int_{3/2}^2 \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}} \right) du \\
&= \int_{3/2}^2 \left(\sqrt{u} - \frac{1}{\sqrt{u}} \right) du = \int_{3/2}^2 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}} \right) du = \left[\frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right]_{3/2}^2 \\
&= \left[\frac{2}{3} u \sqrt{u} - 2\sqrt{u} \right]_{3/2}^2 = \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2} \right) - \left(\sqrt{\frac{3}{2}} - 2\sqrt{\frac{3}{2}} \right) = \sqrt{\frac{3}{2}} - \frac{2\sqrt{2}}{3}
\end{aligned}$$

Important Note. Sometimes in your work you may face integrals $\int_a^b f(x) dx$ where the upper boundary b is actually smaller than the lower boundary a . However, in these cases too the First Fundamental Theorem of Calculus will be applied as usual; see the following example for this case:

Example . Evaluate the integral

$$\int_0^{\frac{1}{2}} \frac{x^3}{1-x^4} dx$$

Solution.

$$\begin{cases} u = 1 - x^4 \\ du = -4x^3 dx \end{cases} \quad \begin{cases} x = \frac{1}{2} \Rightarrow u = \frac{15}{16} \\ x = 0 \Rightarrow u = 1 \end{cases}$$

Then

$$\int_0^{\frac{1}{2}} \frac{x^3}{1-x^4} dx = \int_1^{\frac{15}{16}} \frac{-\frac{1}{4} du}{u} = -\frac{1}{4} \int_1^{\frac{15}{16}} \frac{1}{u} du = -\frac{1}{4} \left[\ln |u| \right]_{u=1}^{u=\frac{15}{16}}$$

$$= -\frac{1}{4} \left\{ \ln\left(\frac{15}{16}\right) - \ln(1) \right\} = -\frac{1}{4} \left\{ \ln\left(\frac{15}{16}\right) - 0 \right\} = -\frac{\ln(\frac{15}{16})}{4}$$