Change of Variable in Indefinite Integral
section 5.3

Note. If $u = f(x)$ is a function of x , then the equality $\frac{du}{dx} = f'(x)$ can be written symbolically in the form $du = f'(x)\,dx$. We call du the "differential of u". Here are some examples:

$$\begin{cases} d(x^2) = 2x\,dx \\ \\ d(\sin x) = \cos x\,dx \\ \\ d\left(t^{\sqrt{3}}\right) = \sqrt{3}\,t^{\sqrt{3}-1}\,dt \\ \\ d\left(\sin(y^2)\right) = 2y\,\cos(y^2)\,dy \end{cases}$$

Note. Since differentiation preserves addition and scalar multiplication , the differential which is a product of differentiation does the same thing:

$$\begin{cases} d(u+v) = du + dv \\ \\ d(c\,u) = c\,du \qquad\qquad c \text{ being a constant} \end{cases}$$

Now we will see in some examples how the method of **change of variable** works:

Example (section 5.3 exercise 24). Evaluate $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx$

Solution. Set $u = 1+\sqrt{x} = 1 + x^{\frac{1}{2}}$. Differentiation gives us

$$du = \frac{1}{2}x^{-\frac{1}{2}}\,dx = \frac{1}{2\sqrt{x}}\,dx \quad \overset{\text{calculate } dx \text{ in terms of } u}{\Longrightarrow} \quad dx = 2\sqrt{x}\,du = 2(u-1)du$$

So:

$$\begin{aligned}
\int \sqrt{1+\sqrt{x}}\,dx &= \int \sqrt{u}(2(u-1))du = 2\int u^{\frac{1}{2}}(u-1)du = 2\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du \\
&= 2\left\{\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right\} + C = \frac{4}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + C \\
&= \frac{4}{5}(1+\sqrt{x})^{\frac{5}{2}} - \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} + C
\end{aligned}$$

Note. When you see an integration of the form $\int \sin^{\alpha} \cos^{\beta}$, where at least one of α or β is an odd number, then we can apply the technique of change of variable to calculate the integral. Here are two examples:

Example. Evaluate $\displaystyle\int \sin^5 x \cos^2 x\,dx$

Solution. Here $\sin x$ appears with an odd-number exponent. We keep one of its exponents together with dx :

$$\int \sin^5 x \cos^2 x\,dx = \int \sin^4 x \cos^2 x \{\sin x\,dx\}$$

$$= \int \sin^4 x \cos^2 x \; d(-\cos x)$$

$$= -\int (1-\cos^2 x)^2\ \cos^2 x\ d(\cos x)$$

Then we make the change of variable $u = \cos x$ and continue:

$$= -\int (1-u^2)^2 u^2\, du$$

$$= -\int (1-2u^2+u^4) u^2\, du$$

$$= -\int (u^2-2u^4+u^6)\, du$$

$$= -\left\{\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\right\} + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C \qquad \checkmark$$

Example. Evaluate $\int \sin^{\frac{3}{2}} x \cos^5 x\, dx$

Solution. Here $\cos x$ appears with an odd-number exponent. We keep one of its exponents together with dx :

$$\int \sin^{\frac{3}{2}} x \cos^5 x\, dx = \int \sin^{\frac{3}{2}} x \cos^4 x \{\cos x\, dx\}$$

$$= \int \sin^{\frac{3}{2}} x \cos^4 x\ d(\sin x)$$

$$= \int \sin^{\frac{3}{2}} x \cos^4 x\ d(\sin x)$$

$$= \int \sin^{\frac{3}{2}} x(1 - \sin^2 x)^2 \; d(\sin x)$$

$$= \int \sin^{\frac{3}{2}} x(1 - \sin^2 x)^2 \; d(\sin x)$$

Then we make the change of variable $u = \sin x$ and continue:

$$= \int u^{\frac{3}{2}}(1 - u^2)^2 \; du$$

$$= \int u^{\frac{3}{2}}(1 - 2u^2 + u^4) \; du$$

$$= \int \left\{ u^{\frac{3}{2}} - 2u^{\frac{7}{2}} + u^{\frac{11}{2}} \right\} \; du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - 2\left(\frac{2}{9}u^{\frac{9}{2}}\right) + \frac{2}{13}u^{\frac{13}{2}} + C$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{9}u^{\frac{9}{2}} + \frac{2}{13}u^{\frac{13}{2}} + C$$

Example (section 5.3 exercise 15). Evaluate $\int \frac{x^3}{(3 - x^2)^3} \, dx$ for $x > 0$.

Solution. Set $u = (3 - x^2)$. Then

$$du = -2x \, dx = -2\left(\sqrt{3 - u}\right) dx \quad \Rightarrow \quad dx = \frac{-du}{2\sqrt{3 - u}}$$

Then

$$\int \frac{x^3}{(3-x^2)^3}\,dx = \int \frac{(3-u)^{\frac{3}{2}}}{u^3}\ \frac{-du}{2\sqrt{3-u}}$$

$$= -\int \frac{3-u}{2u^3}\,du$$

$$= -\int \left(\frac{3}{2u^3} - \frac{1}{2u^2}\right)\,du$$

$$= -\int \left(\frac{3}{2}u^{-3} - \frac{1}{2}u^{-2}\right)\,du$$

$$= -\left\{\frac{3}{2}\left(\frac{u^{-2}}{-2}\right) - \frac{1}{2}\left(\frac{u^{-1}}{-1}\right)\right\} + C$$

$$= -\left\{-\frac{3}{4}u^{-2} + \frac{1}{2}u^{-1}\right\} + C$$

$$= -\left\{-\frac{3}{4u^2} + \frac{1}{2u}\right\} + C$$

$$= \frac{3}{4(3-x^2)^2} - \frac{1}{2(3-x^2)} + C \qquad \checkmark$$

Example (section 5.3 exercise 17). Evaluate $\displaystyle\int \frac{(1+\sqrt{u})^{\frac{1}{2}}}{\sqrt{u}}\,du$

Solution. Set $y = 1 + \sqrt{u} = 1 + u^{\frac{1}{2}}$. Then

$$dy = \frac{1}{2}\,u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}\,du \quad \Rightarrow \quad 2\sqrt{u}\ dy = du \quad \Rightarrow \quad 2(y-1)dy = du$$

Then

$$\int \frac{(1+\sqrt{u})^{\frac{1}{2}}}{\sqrt{u}}\,du$$

$$= \int \frac{y^{\frac{1}{2}}}{y-1}\{2(y-1)\,dy\} = 2\int y^{\frac{1}{2}}\,dy$$

$$= 2\left(\frac{2}{3}y^{\frac{3}{2}}\right) + C$$

$$= \frac{4}{3}(1+\sqrt{u})^{\frac{3}{2}} + C$$