Concavity and Points of Inflection
sections 4.4

Definition. Consider a function f whose derivative exists on an interval I.

- is said to be **concave upward** on an interval I if $f'(x)$ (the slope of the tangent line) increases on I .
- is said to be **concave downward** on an interval I if $f'(x)$ (the slope of the tangent line) decreases on I .
- Points on both sides of which the concavity changes are called **points of inflection**

Note. Figure on page 257 was explained in class.

Question. Is there an easy way of finding the concavity of a function?. Yes , the following therem answers this question.

Theoprem (Test of Concavity). Consider a f on an interval I.

- If $f''(x) \geq 0$ for all points of I , then f is concave up on I.
- If $f''(x) \leq 0$ for all points of I , then f is concave down on I.
- If x_0 is a point of inflection of f , then either $f''(x_0) = 0$ or $f''(x_0)$ does not exist.

Note. Put the following two in comparison:

$$\boxed{\text{If } x_0 \text{ is a point of relative extrema , then either } f'(x_0) = 0 \text{ or } f'(x_0) \text{ does not exist}}$$

$$\boxed{\text{If } x_0 \text{ is a point of inflection , then either } f''(x_0) = 0 \text{ or } f''(x_0) \text{ does not exist}}$$

Example (section 4.4 exercise 1). Determine where the graph of the function $f(x) = x^3 - 3x^2 - 3x + 5$ is concave upward and concave downward , and has points of inflection.

Solution.

$f'(x) = 3x^2 - 6x - 3$

$f''(x) = 6x - 6 = 6(x-1)$

There are no points where f'' does not exist , so the point $x = 1$ is the only candidate for being an inflection point (still we need to check whether this point is an inflection point or not). We use the following table:

		1	
$x-1$	$-$	$\bullet$	$+$
f''	$-$		$+$
f	$\cap$		$\cup$
		inflection point	

The function is concave down on $(-\infty\,,\,1]$ and concave up on $[1\,,\,\infty)$. The only inflection point is $x = 1$.

Example (section 4.4 exercise 3). Determine where the graph of the function $f(x) = x^2 + \frac{1}{x^2}$ is concave upward and concave downward , and has points of inflection.

Solution.

$f(x) = x^2 + x^{-2}$

$f'(x) = 2x - 2x^{-3}$

$f''(x) = 2 + 6x^{-4} = 2 + \frac{6}{x^4} = \frac{2x^4+6}{x^4} > 0$

Note that f'' is always positive , therefore the function is everywhere concave up and there are no inflection points.

Example (section 4.4 exercise 13). Determine where the graph of the function $f(x) = x^2\, e^{3x}$ is concave upward and concave downward , and has points of inflection.

Solution.

$$f'(x) = \{x^2\}'\{\,e^{3x}\} + \{x^2\}\{\,e^{3x}\}'$$
$$f'(x) = \{2x\}\{\,e^{3x}\} + \{x^2\}\{3\,e^{3x}\}$$
$$f'(x) = \{2x + 3x^2\}\{e^{3x}\}$$
$$f''(x) = \{2x + 3x^2\}'\{e^{3x}\} + \{2x + 3x^2\}\{e^{3x}\}'$$
$$f''(x) = \{2 + 6x\}\{e^{3x}\} + \{2x + 3x^2\}\{3\,e^{3x}\}$$
$$f''(x) = \{(2 + 6x) + (6x + 9x^2)\}\{e^{3x}\}$$
$$f''(x) = \{9x^2 + 12x + 2\}\{e^{3x}\}$$
$$f''(x) = 0 \quad \Rightarrow \quad 9x^2 + 12x + 2 = 0 \quad \Rightarrow \quad x = \tfrac{-b\pm\sqrt{b^2-4ac}}{2a} = \tfrac{-12\pm\sqrt{144-72}}{18} = \tfrac{-12\pm\sqrt{72}}{18}$$

$$= \tfrac{-12\pm\sqrt{(36)(2)}}{18} = \tfrac{-12\pm6\sqrt{2}}{18} = \tfrac{-2\pm\sqrt{2}}{3} = \begin{cases} \tfrac{-2-\sqrt{2}}{3} \\ \tfrac{-2+\sqrt{2}}{3} \end{cases}$$

$$\Rightarrow \quad 9x^2 + 12x + 2 = 9\left(x^2 + \tfrac{4}{3}x + \tfrac{2}{9}\right) = 9\left(x - \tfrac{-2-\sqrt{2}}{3}\right)\left(x - \tfrac{-2+\sqrt{2}}{3}\right)$$

		$\frac{-2-\sqrt{2}}{3}$		$\frac{-2+\sqrt{2}}{3}$	
$x - \frac{-2-\sqrt{2}}{3}$	$-$	$\bullet$	$+$		$+$
$x - \frac{-2+\sqrt{2}}{3}$	$-$		$-$	$\bullet$	$+$
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$
		inflection point		**inflection point**	

Note that there are no points where f'' does not exist , therefore the points $\frac{-2+\sqrt{2}}{3}$ and $\frac{-2-\sqrt{2}}{3}$ are the only inflection points.

Example. Consider the function $f(x) = 3x^4 - 8x^3$. Answer the following parts:

(i) Find the intervals on which f is increasing and decreasing .

(ii) Find the relative extrema of f and the points that give them.

(iii) Find the intervals on which f is concave upward and concave downward.

(iv) What are the points of inflection ?

Solution.

$f(x) = 3x^4 - 8x^3$

$f'(x) = 12x^3 - 24x^2$

$f'(x) = 0 \quad \Rightarrow \quad x = 0\ ,\ 2$

There an no points where f' does not exist , so the points $x = 0\ ,\ 2$ are the only critical points.

		0		2	
$x-2$	$-$		$-$	$\bullet$	$+$
f'	$-$		$-$		$+$
f	$\searrow$		$\searrow$		$\nearrow$
				relative min	

The function is decreasing on $(-\infty\,,\,2]$ and is increasing on $[2\,,\,\infty)$

The point $x = 2$ gives the relative minimum. The value of relative minimum is $f(2) = -16$

So far we have answered parts **(i)** and **(ii)**. To answer parts **(iii)** and **(iv)** we need f''.

$f''(x) = 36x^2 - 48x = 12x(3x-4)$

$f''(x) = 0 \quad \Rightarrow \quad x = 0\ ,\ \frac{4}{3}$

There are no pints where f'' does not exist , otherwise we would consider them as candidates of being inflection points.

		0		$\frac{4}{3}$	
x	$-$	●	$+$		$+$
$3x-4$	$-$		$-$	●	$+$
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$
		inflection point		**inflection point**	

The function is concave up on the intervals $(-\infty\,,\,0]$ and $[\frac{4}{3}\,,\,\infty)$, and it is concave down on the interval $[0\,,\,\frac{4}{3}]$. The points $x = 0$ and $x = \frac{4}{3}$ are the only pints of inflection.