# Concavity and Points of Inflection sections 4.4

**Definition**. Consider a function f whose derivative exists on an interval I.

- is said to be **concave upward** on an interval I if f'(x) (the slope of the tangent line) increases on I.
- is said to be **concave downward** on an interval I if f'(x) (the slope of the tangent line) decreases on I.
- Points on both sides of which the concavity changes are called **points of inflection**

Note. Figure on page 257 was explained in class.

**Question**. Is there an easy way of finding the concavity of a function?. Yes , the following therem answers this question.

**Theoprem (Test of Concavity)**. Consider a f on an interval I.

- If  $f''(x) \ge 0$  for all points of I, then f is concave up on I.
- If  $f''(x) \leq 0$  for all points of I, then f is concave down on I.
- If  $x_0$  is a point of inflection of f, then either  $f''(x_0) = 0$  or  $f''(x_0)$  does not exist.

<u>Note</u>. Put the following two in comparison:

If  $x_0$  is a point of relative extrema, then either  $f'(x_0) = 0$  or  $f'(x_0)$  does not exist

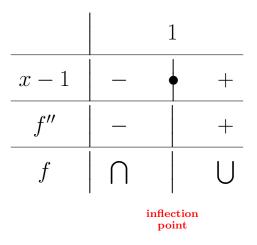
If  $x_0$  is a point of inflection, then either  $f''(x_0) = 0$  or  $f''(x_0)$  does not exist

Example (section 4.4 exercise 1). Determine where the graph of the function  $f(x) = x^3 - 3x^2 - 3x + 5$  is concave upward and concave downward, and has points of inflection.

### Solution.

$$f'(x) = 3x^2 - 6x - 3$$
$$f''(x) = 6x - 6 = 6(x - 1)$$

There are no points where f'' does not exist, so the point x = 1 is the only candidate for being an inflection point (still we need to check whether this point is an inflection point or not). We use the following table:



The function is concave down on  $(-\infty, 1]$  and concave up on  $[1, \infty)$ . The only inflection point is x = 1.

Example (section 4.4 exercise 3). Determine where the graph of the function  $f(x) = x^2 + \frac{1}{x^2}$  is concave upward and concave downward , and has points of inflection.

## Solution.

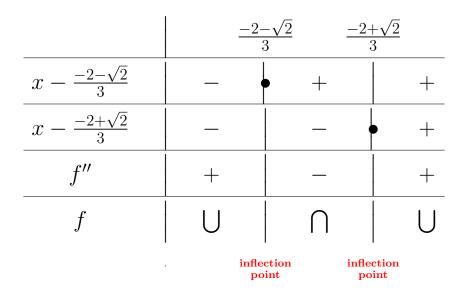
$$\begin{split} f(x) &= x^2 + x^{-2} \\ f'(x) &= 2x - 2x^{-3} \\ f''(x) &= 2 + 6x^{-4} = 2 + \frac{6}{x^4} = \frac{2x^4 + 6}{x^4} > 0 \end{split}$$

Note that f'' is always positive , therefore the function is everywhere concave up and there are no inflection points.

Example (section 4.4 exercise 13). Determine where the graph of the function  $f(x) = x^2 e^{3x}$  is concave upward and concave downward, and has points of inflection.

# Solution.

$$\begin{aligned} f'(x) &= \{x^2\}'\{e^{3x}\} + \{x^2\}\{e^{3x}\}'\\ f'(x) &= \{2x\}\{e^{3x}\} + \{x^2\}\{3e^{3x}\}\\ f'(x) &= \{2x + 3x^2\}\{e^{3x}\} + \{2x + 3x^2\}\{e^{3x}\}'\\ f''(x) &= \{2x + 3x^2\}'\{e^{3x}\} + \{2x + 3x^2\}\{3e^{3x}\}\\ f''(x) &= \{2 + 6x\}\{e^{3x}\} + \{2x + 3x^2\}\{3e^{3x}\}\\ f''(x) &= \{(2 + 6x) + (6x + 9x^2)\}\{e^{3x}\}\\ f''(x) &= \{9x^2 + 12x + 2\}\{e^{3x}\}\\ f''(x) &= \{9x^2 + 12x + 2\}\{e^{3x}\}\\ f''(x) &= 0 \implies 9x^2 + 12x + 2 = 0 \implies x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-12\pm\sqrt{144-72}}{18} = \frac{-12\pm\sqrt{72}}{18}\\ &= \frac{-12\pm\sqrt{(36)(2)}}{18} = \frac{-12\pm6\sqrt{2}}{18} = \frac{-2\pm\sqrt{2}}{3} = \begin{cases} \frac{-2-\sqrt{2}}{3}\\ \frac{-2+\sqrt{2}}{3} \end{cases}\\ \Rightarrow \quad 9x^2 + 12x + 2 = 9\left(x^2 + \frac{4}{3}x + \frac{2}{9}\right) = 9\left(x - \frac{-2-\sqrt{2}}{3}\right)\left(x - \frac{-2+\sqrt{2}}{3}\right) \end{aligned}$$



Note that there are no points where f'' does not exist, therefore the points  $\frac{-2+\sqrt{2}}{3}$  and  $\frac{-2-\sqrt{2}}{3}$  are the only inflection points.

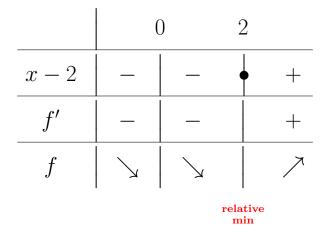
**Example**. Consider the function  $f(x) = 3x^4 - 8x^3$ . Answer the following parts:

- (i) Find the intervals on which f is increasing and decreasing .
- (ii) Find the relative extrema of f and the points that give them.
- (iii) Find the intervals on which f is concave upward and concave downward.
- (iv) What are the points of inflection ?

### Solution.

$$f(x) = 3x^4 - 8x^3$$
  
$$f'(x) = 12x^3 - 24x^2$$
  
$$f'(x) = 0 \quad \Rightarrow \quad x = 0 \ , \ 2$$

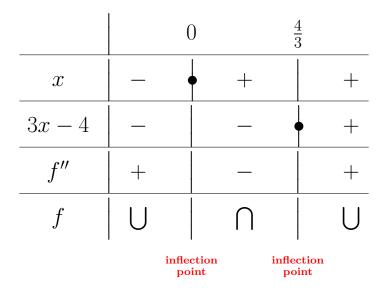
There an no points where f' does not exist, so the points x = 0, 2 are the only critical points.



The function is decreasing on  $(-\infty, 2]$  and is increasing on  $[2, \infty)$ The point x = 2 gives the relative minimum. The value of relative minimum is f(2) = -16So far we have answered parts (i) and (ii). To answer parts (iii) and (iv) we need f''.

$$f''(x) = 36x^2 - 48x = 12x(3x - 4)$$
  
$$f''(x) = 0 \implies x = 0, \frac{4}{3}$$

There are no pints where f'' does not exist , otherwise we would consider them as candidates of being inflection points.



The function is concave up on the intervals  $(-\infty, 0]$  and  $[\frac{4}{3}, \infty)$ , and it is concave down on the interval  $[0, \frac{4}{3}]$ . The points x = 0 and  $x = \frac{4}{3}$  are the only pints of inflection.