Continuity

section 2.4

Definition: Let a be a point of the domain of a function f. Then f is said to be continuous at a provided that the limit $\lim_{x\to a} f(x)$ exists and we have $\lim_{x\to a} f(x) = f(a)$. Geometrically, there is no hole in the graph of the function at the point $x = a$. Some important examples of functions that are continuous on their domains are: polynomials , trigonometric function , exponential functions a^x, the log functions $\log_a(x)$, the power functions x^b, the root functions $\sqrt[n]{x^m}$, the absolute value function $|x|$.

Example: Let k be any real number. Find the value of k such that the function

$$f(x) = \begin{cases} 2 + kx & x \geq 1 \\ 2k + 3x^2 & x < 1 \end{cases}$$

is continuous at $x = 1$.

Solution: We have $f(1) = 2 + k$ .For the continuity to hold we must have both

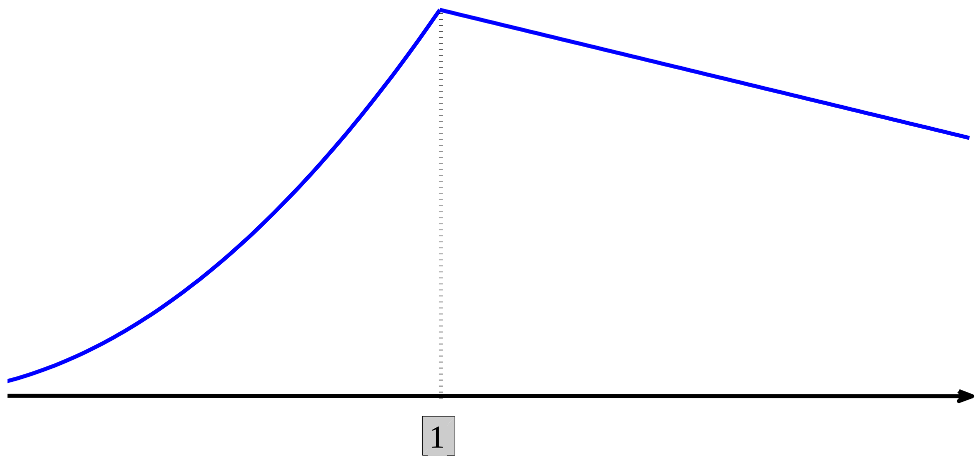
$$\begin{cases} \lim_{x\to 1^+} f(x) = f(1) \\ \lim_{x\to 1^-} f(x) = f(1) \end{cases}$$

The requirement $\lim_{x\to 1^+} f(x) = f(1)$ is equivalent to $\lim_{x\to 1^+} 2+kx = 2+k$ which is the same as the equality $2+k = 2+k$ which gives nothing. But the requirement $\lim_{x\to 1^-} f(x) = f(1)$ is equivalent to $\lim_{x\to 1^-} 2k + 3x^2 = 2 + k$ which is the same as $2k + 3 = 2 + k$ resulting in $\boxed{k = -1}$.

Note: For $k = -1$ this function becomes

$$f(x) = \begin{cases} 2 - x & x \geq 1 \\ -2 + 3x^2 & x < 1 \end{cases}$$

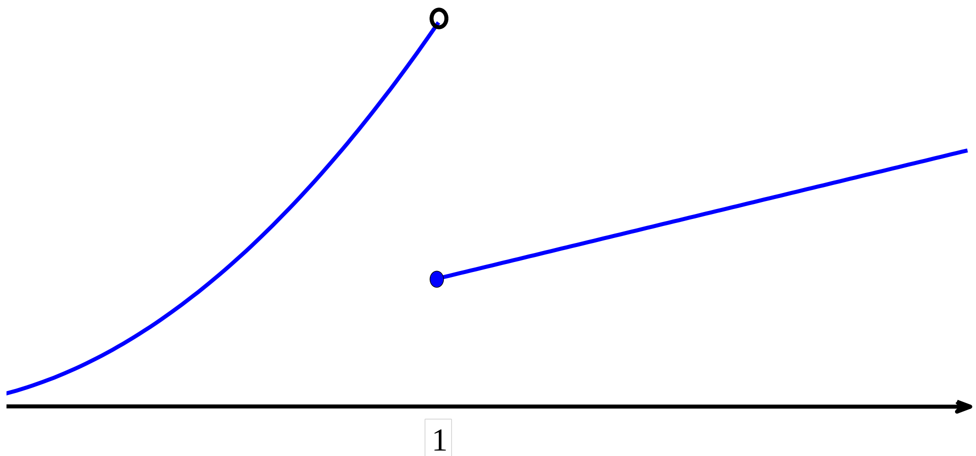
whose graph is



For $k = 1$ the function becomes

$$f(x) = \begin{cases} 2 + x & x \geq 1 \\ 2 + 3x^2 & x < 1 \end{cases}$$

whose graph is



Example: Find the values of a and b such that the function

$$f(x) = \begin{cases} a(x^2+1) - 2bx & x < 3 \\ 4 & x = 3 \\ 2a + 3bx & x > 3 \end{cases}$$

is continuous everywhere.

Solution: The function is continuous on the interval $(-\infty\,,\,3)$ because on this interval it is the same as the polynomial $a(x^2+1)-2bx$ and we know that the polynomials are continuous. The function is continuous on the interval $(3\,,\,\infty)$ because it a polynomial there, namely $2a+3bx$. The only point at which continuity is not guaranteed is the point $x = 3$ at which the rule of the function changes and the function behaves differently on both sides of this point. Now

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \Big\{a(x^2+1) - 2bx\Big\} = 10a - 6b$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \Big\{2a + 3bx\Big\} = 2a + 9b$$

For this function to be continuous at $x = 3$ we must have

$$\begin{cases} \lim\limits_{x\to 3^-} f(x) = f(3) \\ \lim\limits_{x\to 3^+} f(x) = f(3) \end{cases} \Rightarrow \begin{cases} 10a - 6b = 4 \\ 2a + 9b = 4 \end{cases} \overset{\text{simplifying}}{\Rightarrow} \begin{cases} 5a - 3b = 2 \\ 2a + 9b = 4 \end{cases}$$

To solve this system, multiply the first row by 3 and then add it to the second row to get $17a = 10$ resulting in $\boxed{a = \frac{10}{17}}$. By putting this value of a into one of the equations one gets $\boxed{b = \frac{16}{51}}$.

Definition: Let I be an interval in the real line. A function $f : I \to \mathbb{R}$ is said to be continuous on I if it is continuous at all points of I ; geometrically, there is no hole on the graph anywhere on I.

♣ **Theorem (algebraic properties of continuous functions)**: Suppose that both f and g are continuous at a point a. Then the following functions are continuous at a too:

$$f \pm g \quad , \quad fg \quad , \quad cf \quad \text{where c is a constant}$$

Also the function $\frac{f}{g}$ is continuous at a provided that $g(a) \neq 0$.

Proof: We prove the continuity of $\frac{f}{g}$ at a under the assumption that $g(a) \neq 0$. In fact, due to algebraic properties of the "limit" , we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{f(a)}{g(a)} \quad \Rightarrow \quad \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} \quad \overset{\text{by definition}}{\Longrightarrow} \quad \frac{f}{g} \text{ is continuous at } a$$

Note: Similar theorems hold for the "continuity from the left" and "continuity from the right"

♣ **Theorem (another theorem on algebraic properties of continuous functions)**: Suppose that both f and g are continuous on their domains. Then the functions $f \pm g$, fg , cf , and $\frac{f}{g}$ are continuous on their domains.

Note: Similar theorems hold for the "continuity from the left" and "continuity from the right"

♣ **Theorem (composition of continuous functions)**:

i) If f is continuous at a and if g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

ii) If f is continuous on its domain and g is continuous on its domain, then $g \circ f$ is continuous on its domain.

Example: The function $h(x) = \sqrt[3]{x^2 - x + 1}$ is a composition of continuous function and hence is continuous on its domain. In fact it is the composition $g \circ f$ of the two continuous functions $f(x) = x^2 - x + 1$ and $g(x) = \sqrt[3]{x}$.

Example: The function

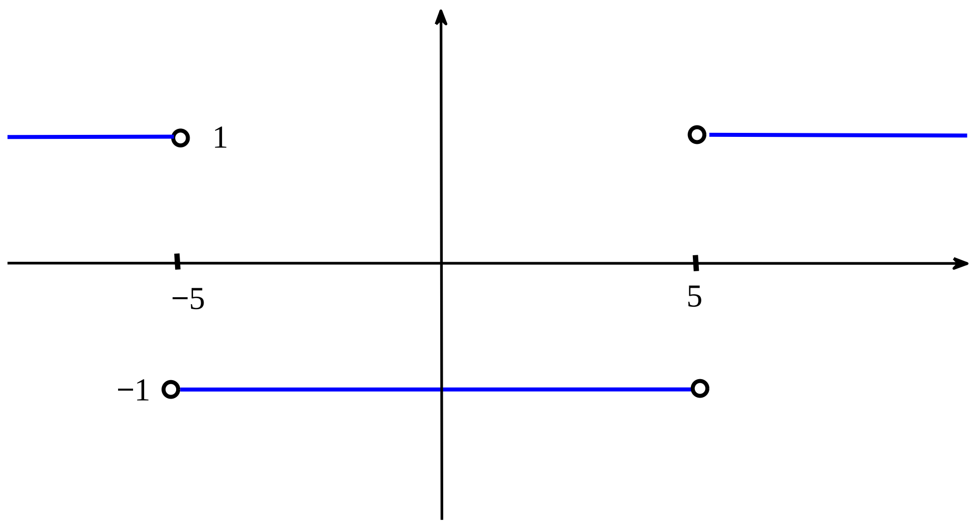
$$f(x) = \frac{\tan\left(\sqrt[4]{\frac{\cos(x+1)}{x^2-3}}\right)}{\sin(2x+1)}$$

is the result of compositions and algebraic operations (addition, subtraction, multiplication, and division) on continuous functions, therefore is a continuous function on its domain.

Example (from the textbook): The function

$$f(x) = \frac{|x^2 - 25|}{x^2 - 25}$$

is the quotient of two functions which are continuous on their domains, therefore the function f is continuous on its domain. The graph of this function is



Although there are breaks on the graph at the point ± 5 , however these points are not in the domain of the function f therefore this function is continuous everywhere.

<u>Explanation about the graph on the top portion of page 129 of the textbook</u>: The points b , d , and e are not in the domain, therefore there is no discontinuity at those points. But, the function is discontinuous at points a and c. We have two-sided discontinuity at a , while the discontinuity at c is only on the right-hand side.

♣ **<u>Theorem (transmission property of continuous functions)</u>**: If f is continuous at L and if $\lim_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f(g(x)) = f(L)$$

equivalently

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

<u>Note</u>: This theorem is true for the limits $\lim_{x\to a^+}$, $\lim_{x\to a^-}$, $\lim_{x\to\infty}$, and $\lim_{x\to -\infty}$ too.

Example: Evaluate the limit $\lim\limits_{x\to-\infty} \sin(\sqrt{x^2+x+1}-\sqrt{x^2-x+1})$.

Solution:

$$\lim_{x\to-\infty} \sin(\sqrt{x^2+x+1}-\sqrt{x^2-x+1}) = \sin\left(\lim_{x\to-\infty} (\sqrt{x^2+x+1}-\sqrt{x^2+x-1})\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{(\sqrt{x^2+x+1}-\sqrt{x^2-x+1})(\sqrt{x^2+x+1}+\sqrt{x^2-x+1})}{(\sqrt{x^2+x+1}+\sqrt{x^2-x+1})}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{(x^2+x+1)-(x^2-x+1)}{\sqrt{x^2+x+1}+\sqrt{x^2-x+1}}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{2x}{\sqrt{x^2+x+1}+\sqrt{x^2-x+1}}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{2x}{\sqrt{x^2(1+\frac{1}{x}+\frac{1}{x^2})}+\sqrt{x^2(1-\frac{1}{x}+\frac{1}{x^2})}}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{2x}{|x|\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+|x|\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{2x}{(-x)\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+(-x)\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{2}{(-1)\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+(-1)\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{2}{(-1)\sqrt{1+0+0}+(-1)\sqrt{1-0+0}}\right) = \sin(-1) \qquad \checkmark$$

♣ **Theorem** : Suppose that $f : I \to \mathbb{R}$ is a continuous one-to-one function on an interval I. Then

i) The image of I under f is an interval; let us call it J.

ii) The inverse function f^{-1} is continuous on J

Note: We don't see any application of this theorem for now. This theorem is mainly used for theoretical purposes.

♣ **Theorem**: If f is continuous at a and g is continuous at $f(a)$ then $g \circ f$ is continuous at a.