Differentiation
(sections 3.1 to 3.4)

We recall from high school that one needs the slope m and one point (x_0, y_0) of a non-vertical line $(m \neq \pm\infty)$ in order to write the equation of it. The equation of the line is

$$y - y_0 = m(x - x_0) \qquad (*)$$

If (x_1, y_1) and (x_2, y_2) are two points of the line , then $m = \frac{y_2 - y_1}{x_2 - x_1}$ and either of these two points can serve as the point (x_0, y_0) . If for instance we take (x_1, y_1) as the substitute for (x_0, y_0) , then the equation takes on a new form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

equivalently

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

which is easier to memorize.

Example: Find the equation of the line which passes through two points $(1, -1)$ and $(0, 2)$.

Solution:

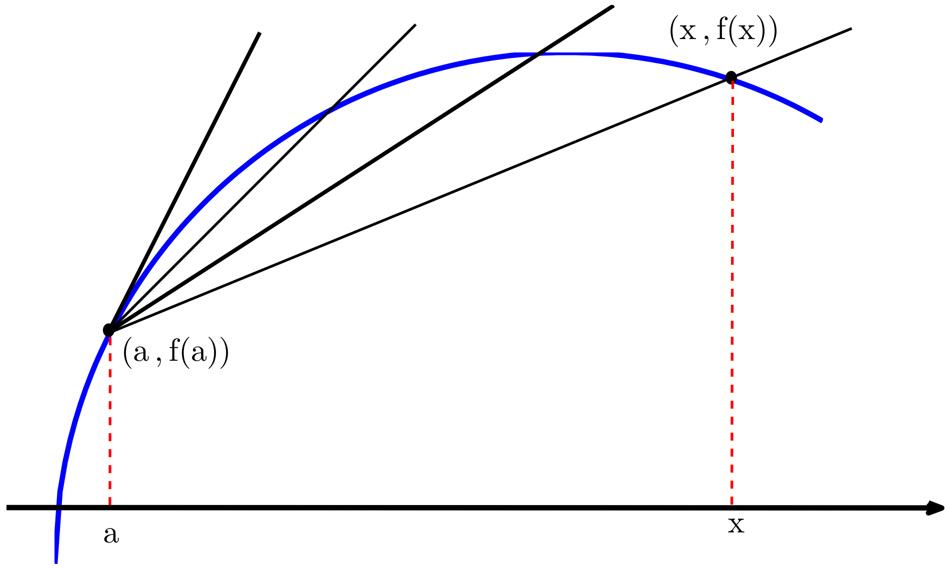
$$\frac{y - (-1)}{x - (1)} = \frac{(2) - (-1)}{(0) - (1)} \qquad \Rightarrow \qquad \frac{y + 1}{x - 1} = \frac{3}{-1} \qquad \Rightarrow \qquad y + 1 = -3(x - 1) \qquad \Rightarrow \qquad y = -3x + 2$$

Consider a function $y = f(x)$ and the tangent line T at some point $(a\,,\, f(a))$ on the graph. Imagine another point $(x\,,\, f(x))$ on the graph and call L the line joining $(a\,,\, f(a))$ and $(x\,,\, f(x))$. The slope of the line L is $\frac{f(x) - f(a)}{x - a}$. As x approaches a the line L will approach the line T and therefore the slope of L approaches that of T. So in the limit we have:

$$\text{slope of } T \quad = \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

In other words:

$$\text{slope of the tangent line at } a \quad = \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



In addition to this application of giving the slope of the tangent line , the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ has some other important applications. Therefore we give it a name:

Definition: If for a function f and a point a in its domain the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, then we call this limit the **derivative** of the function f at point a and we may denote it by $f'(a)$ briefly.

$$f'(a) \overset{\text{definition}}{=} \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

When this limit exists, we may also say that the function f is **differentiable** at a.

Note: If $x \to a$, then $x - a \to a - a = 0$. The **increment** $x - a$ may be denoted by h or Δx. In the textbook it is denoted by h. By letting $x - a = h$ we have $x = a + h$ and therefore the expression $\frac{f(x)-f(a)}{x-a}$ can be equivalently written as $\frac{f(a+h)-f(a)}{h}$, and the equality $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ can be equivalently written as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

If Δx is used instead of h, then we have

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

Note: The difference $f(x) - f(a)$ is indeed the increment in the values of the function and it might be written as Δf or Δy. So , with this notation, we have

$$f'(a) = \lim_{h \to 0} \frac{\Delta f}{\Delta x} \qquad \text{at base point } a$$

Note that

$$\Delta f = f(x) - f(a) \qquad \Longrightarrow \qquad f(x) = f(a) + \Delta f$$

new value of f = old value of f plus increment in f

Example: Given two functions f and g and a base point a we have

$$\begin{aligned} \Delta(fg) &= (fg)(x) - (fg)(a) \\ &= f(x)g(x) - f(a)g(a) \\ &= \{f(a) + \Delta f\}\{g(a) + \Delta g\} - f(a)g(a) \\ &= f(a)\Delta g + g(a)\Delta f + (\Delta f)(\Delta g) \end{aligned}$$

Exercise: Find expressions for $\Delta(f \pm g)$ and $\Delta(cf)$ and $\Delta\left(\frac{f}{g}\right)$ in terms of Δf and Δg.

Convention: The derivative of a function $y = y(x)$ at all possible points x where the derivative exists is denoted by $\frac{dy}{dx}$. If the derivative at a particular point $x = a$ is being considered, then we denote it by $\left.\frac{dy}{dx}\right|_{x=a}$

Example: Calculate Δy for the function $y = \frac{1}{2x-1}$ at an arbitrary point x. What is the value of Δy at the base point $x = 1$?. Finally, find the derivative $\frac{dy}{dx}$ at the arbitrary point x in the domain of $y(x)$. what is the value of the derivative at the point $x = 1$. See also Example 3.10 on page 164 of the textbook.

Solution: **Step 1**:

$$\Delta y = y(x+\Delta x) - y(x) = \frac{1}{2(x+\Delta x)-1} - \frac{1}{2x-1} = \frac{\{2x-1\} - \{2(x+\Delta x)-1\}}{\{2(x+\Delta x)-1\}\{2x-1\}}$$

$$= \frac{-2\Delta x}{\{2(x+\Delta x)-1\}\{2x-1\}}$$

Step 2.

$$y'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{-2\Delta x}{\{2(x+\Delta x)-1\}\{2x-1\}}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-2}{\{2(x+\Delta x)-1\}\{2x-1\}} = \frac{-2}{(2x-1)^2}$$

So,

$$\frac{d}{dx}\left(\frac{1}{2x-1}\right) = \frac{-2}{(2x-1)^2}$$

or we may write it as

$$\left(\frac{1}{2x-1}\right)' = \frac{-2}{(2x-1)^2}$$

At the particular point $x = 1$ we have

$$\frac{d}{dx}\left(\frac{1}{2x-1}\right)\bigg|_{x=1} = \frac{-2}{(2x-1)^2}\bigg|_{x=1} = \frac{-2}{(2-1)^2} = -2$$

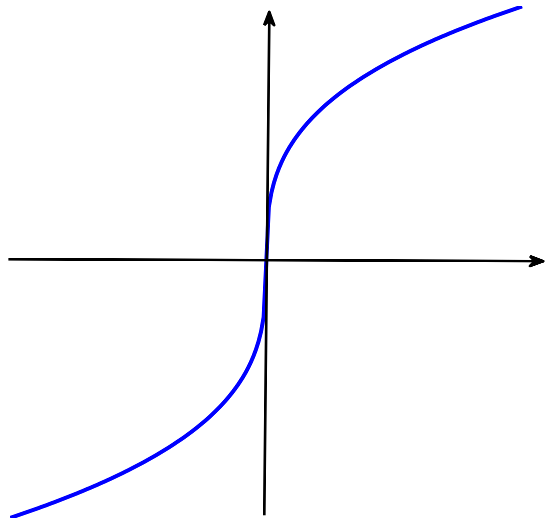
Example: Find the derivative of the function $y = \sqrt[3]{x}$ at the origin.

Solution:

$$y'(0) = \lim_{h \to 0} \frac{y(0+h) - y(0)}{h} = \lim_{h \to 0} \frac{y(h) - y(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h} - 0}{h}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt[3]{h^2}} = \frac{1}{0^+} = \infty \qquad \Longrightarrow \qquad \text{the derivative } \; y'(0) \text{ does not exist}$$

Geometrically , the tangent line is vertical at the origin; see the graph.



Theorem: Differentiability implies continuity , that is, if f is differentiable at a then f is continuous at a.

This is Theorem 3.6 of the textbook.

Proof: By assuming that $f'(a)$ exists we want to show that f is continuous at a , equivalently we want to show that $\lim_{x\to a} f(x) = f(a)$. Equivalently we must show that $\lim_{x\to a} \Big\{ f(x) - f(a) \Big\} = 0$. Here is how:

$$\lim_{x\to a} \Big\{ f(x) - f(a) \Big\} = \lim_{x\to a} \left\{ \frac{f(x) - f(a)}{x - a} \right\} (x - a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \ \lim_{x\to a} (x - a) = f'(a) \, \mathbf{times} \, 0 = 0$$

Corollary: If f is discontinuous at a , then f is not differentiable at a.

Note: The converse to the above theorem is not true. For this, take the function $y(x) = |x|$. Note that the graph of this function has a "cusp" at the origin. This causes the derivative not to exist there. In fact at the point $a = 0$ we have:

$$\lim_{h\to 0} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0} \frac{y(h)-y(0)}{h} = \lim_{h\to 0} \frac{|h|-0}{h} = \begin{cases} \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1 \\ \\ \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{-h}{h} = -1 \end{cases}$$

$\Longrightarrow$ the limit does not exist $\Longrightarrow$ the function is not differentiable at the origin

Note: Geometrically, if there is cusp on the graph of a function, then the function is not differentiable at that point.

The left-hand derivative and right-hand derivative are define by:

$$f'_-(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$$

$$f'_+(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$$

Theorem: For $f'(a)$ to exist it is necessary and sufficient that these conditions are met:

a) both $f'_-(a)$ and $f'_+(a)$ exist

b) $f'_-(a) = f'_+(a)$

Furthermore , if these conditions are met, then the derivative $f'(a)$ equals the common value of $f'_-(a)$ and $f'_+(a)$:

$$f'(a) = f'_-(a) = f'_+(a)$$

Example: For the function $f(x) = |x-3| + x^2$ calculate both $f'_-(3)$ and $f'_+(3)$ and check whether this function is differentiable at $x = 3$.

Solution:

- $f'_{-}(3) = \lim_{h \to 0^-} \dfrac{f(3+h) - f(3)}{h} = \lim_{h \to 0^-} \dfrac{\left\{ |h| + (3+h)^2 \right\} - \{9\}}{h}$

$= \lim_{h \to 0^-} \dfrac{\left\{ |h| + (9 + 6h + h^2) \right\} - \{9\}}{h} = \lim_{h \to 0^-} \dfrac{|h| + 6h + h^2}{h}$

$= \lim_{h \to 0^-} \dfrac{(-h) + 6h + h^2}{h} = \lim_{h \to 0^-} \frac{5h+h^2}{h}$ of indeterminate form $\frac{0}{0}$

$= \lim_{h \to 0^-} (5 + h) = 5$

- $f'_{+}(3) = \lim_{h \to 0^+} \dfrac{f(3+h) - f(3)}{h} = \lim_{h \to 0^+} \dfrac{\left\{ |h| + (3+h)^2 \right\} - \{9\}}{h}$

$= \lim_{h \to 0^+} \dfrac{\left\{ |h| + (9 + 6h + h^2) \right\} - \{9\}}{h} = \lim_{h \to 0^+} \dfrac{|h| + 6h + h^2}{h}$

$= \lim_{h \to 0^+} \dfrac{(h) + 6h + h^2}{h} = \lim_{h \to 0^+} \dfrac{7h + h^2}{h}$ of indeterminate form $\dfrac{0}{0}$

$= \lim_{h \to 0^+} (7 + h) = 7$

$f'_{-}(3) \neq f'_{+}(3) \implies$ it is not differentiable at $x = 3$

Example: Find the derivative of the function

$$f(x) = \begin{cases} 1 - x^2 & x < 1 \\ 0 & x = 1 \\ x - 1 & x > 1 \end{cases}$$

at the point $x = 1$ if exists at all.

Solution: Since the function is defined by different rules on both sides of the point $x = 1$, we have to calculate $f'_-(1)$ and $f'_+(1)$ in order to find out whether $f'(1)$ exists .

- $$f'_-(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{\{1 - (1+h)^2\} - \{0\}}{h}$$

$$= \lim_{h \to 0^-} \frac{1 - (1 + 2h + h^2)}{h} = \lim_{h \to 0^-} \frac{-2h - h^2}{h} \qquad \text{of indeterminate form } \frac{0}{0}$$

$$= \lim_{h \to 0^-} (-2 - h) = -2$$

- $$f'_+(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{\{(1+h) - 1\} - \{0\}}{h}$$

$$= \lim_{h \to 0^+} \frac{h}{h} \qquad \text{of indeterminate form } \frac{0}{0}$$

$$= \lim_{h \to 0^+} (1) = 1$$

$f'_-(1) \neq f'_+(1) \qquad \Longrightarrow \qquad$ it is not differentiable at $x = 1$

Theorem (algebraic properties of differentiation): Suppose that functions f and g are differentiable at a. Then The function $f \pm g$, fg , cf (c being a constant) are differentiable at a. The function $\frac{f}{g}$ is differentiable at a provided that $g(a) \neq 0$. Furthermore we have:

- $(f \pm g)' = f' \pm g'$ at point a
- $(cf)' = c\, f'$ at point a
- $(fg)' = f'g + fg'$ at point a
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ at point a

This theorem is comprised of theorems 3.4 , 3.5, 3.7, and 3.8 of the textbook.

Proof: Here is the proof for the equality $(fg)' = f'g + fg'$ (the other parts of the theorem can be proven with less or more effort)

Step 1. Since g is assumed to be differentiable at a it is continuous at a , therefore $\lim_{\Delta x \to 0} g(x) = g(a)$, equivalently $\lim_{\Delta x \to 0}\{g(x) - g(a)\} = 0$, i.e. $\lim_{\Delta x \to 0} \Delta g = 0$.

Step 2. As we have seen above , at a base point a we have

$$\Delta(fg) = f(a)\Delta g + g(a)\Delta f + (\Delta f)(\Delta g)$$

Now divide both sides by Δx and let $\Delta x \to 0$ to get:

$$
\begin{aligned}
(fg)'(a) &= \lim_{\Delta x \to 0} \tfrac{\Delta(fg)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left\{ f(a)\tfrac{\Delta g}{\Delta x} + g(a)\tfrac{\Delta f}{\Delta x} + \left(\tfrac{\Delta f}{\Delta x}\right)(\Delta g) \right\} \\
&= f(a) \lim_{\Delta x \to 0} \tfrac{\Delta g}{\Delta x} + g(a) \lim_{\Delta x \to 0} \tfrac{\Delta f}{\Delta x} + \lim_{\Delta x \to 0} \left(\tfrac{\Delta f}{\Delta x}\right) \lim_{\Delta x \to 0} (\Delta g) \\
&= f(a)\, g'(a) + g(a) f'(a) + f'(a) \textbf{ times } 0 \\
&= f(a)\, g'(a) + g(a) f'(a)
\end{aligned}
$$

Theorem: The derivative of a constant function is zero, that is, if $f(x) = c$ for all x then $f'(x) = 0$ at all x

Proof:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

Theorem: For $f(x) = x$ we have $f'(x) = 1$ at all x's

Proof:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

Theorem: If n is an arbitrary positive integer (n=1,2,...) , then $\frac{d(x^n)}{dx} = nx^{n-1}$ at all x's ; we call this rule the **power rule for positive integers**.

Proof: We have seen this for $n = 1$ in the previous theorem. Now we proceed to prove it by Mathematical induction. So suppose that we have it for an n :

$$\text{assumption}: \quad \{x^n\}' = nx^{n-1}$$

Now we must prove it for $n+1$ instead of n :

$$\text{assumption :} \quad \{x^{n+1}\}' = (n+1)x^n$$

Here is how:

$$\begin{aligned}
\{x^{n+1}\}' &= \{x^n x\}' \\
&= \{x^n\}'\{x\} + \{x^n\}\{x\}' \\
&= \{nx^{n-1}\}\{x\} + \{x^n\}\{1\} \qquad \text{using the mathematical induction assumption} \\
&= nx^n + x^n = (n+1)x^n \quad \checkmark
\end{aligned}$$

Example: Find the derivative function for the function $y = 2x^3 - x^2 + 1$.

Solution:

$$y' = (2)(3)x^2 - (2)x^1 + 0 = 6x^2 - 2x \quad \checkmark \qquad \text{this is a function of } x$$

Example): Differentiate the function $y = (x^3+2x+1)(x^4-2x^2+x-3)$; simplify your answer.

Solution: Although one can do the expansion first and then find the derivative , but it is easier to apply the product rule :

$$\begin{aligned}
y' &= \{x^3+2x+1\}'\{x^4-2x^2+x-3\} + \{x^3+2x+1\}\{x^4-2x^2+x-3\}' \\
&= \{3x^2+2\}\{x^4-2x^2+x-3\} + \{x^3+2x+1\}\{4x^3-4x+1\} \\
&= \{3x^6-6x^4+3x^3-9x^2+2x^4-4x^2+2x-6\} + \{4x^6-4x^4+x^3-8x^4-8x^2+2x+4x^3-4x+1\} \\
&= \{3x^6-4x^4+3x^3-13x^2+2x-6\} + \{4x^6-12x^4+5x^3-8x^2-2x+1\} \\
&= 7x^6-16x^4+8x^3-21x^2-5
\end{aligned}$$

Example (section 3.4 exercise 12): Find $f'(x)$ in simplified form if

$$f(x) = \frac{x^2+2x+3}{x^2-5x+1}$$

Solution: Apply the quotient rule to get:

$$f'(x) = \frac{\{x^2+2x+3\}'\{x^2-5x+1\} - \{x^2-5x+1\}'\{x^2+2x+3\}}{(x^2-5x+1)^2}$$

$$= \frac{\{2x+2\}\{x^2-5x+1\} - \{2x-5\}\{x^2+2x+3\}}{(x^2-5x+1)^2}$$

$$= \frac{\{2x^3-10x^2+2x+2x^2-10x+2\} - \{2x^3+4x^2+6x-5x^2-10x-15\}}{(x^2-5x+1)^2}$$

$$= \frac{\{2x^3-8x^2-8x+2\} - \{2x^3-x^2-4x-15\}}{(x^2-5x+1)^2} = \frac{-7x^2-4x+17}{(x^2-5x+1)^2} \qquad \checkmark$$

Theorem (power rule for the exponents of the form $\frac{1}{n}$): If n is an arbitrary positive integer (n=1,2,...) , then the derivative of function $y = \sqrt[n]{x} = x^{\frac{1}{n}}$ is $y' = \frac{1}{n}x^{\frac{1}{n}-1}$ i.e. the power rule holds for these functions too.

$$y = x^{\frac{1}{n}} \qquad \Longrightarrow \qquad y' = \frac{1}{n}x^{\frac{1}{n}-1}$$

Example:

$$\left\{x^{\frac{1}{10}}\right\}' = \frac{1}{10}x^{\frac{1}{10}-1} = \frac{1}{10}x^{-\frac{9}{10}} = \frac{1}{10\ x^{\frac{9}{10}}} = \frac{1}{10\ \sqrt[10]{x^9}}$$

Proof of the theorem: Set $a = (x+h)^{\frac{1}{n}}$ and $b = x^{\frac{1}{n}}$. Then by raising to the power of n we will have:

$$\begin{cases} a^n = x+h \\ b^n = x \end{cases} \qquad \Rightarrow \qquad a^n - b^n = h$$

Note that from the continuity of the root function we actually have

$$a \to b \quad \text{when} \quad h \to 0$$

To show $y' = \frac{1}{n}x^{\frac{1}{n}-1}$ we must show that

$$\lim_{h\to 0} \frac{y(x+h)-y(h)}{h} = \frac{1}{n}x^{\frac{1}{n}-1}$$

This is how it is done:

$$y'(x) = \lim_{h\to 0}\frac{y(x+h)-y(x)}{h} = \lim_{h\to 0}\frac{(x+h)^{\frac{1}{n}}-(x)^{\frac{1}{n}}}{h} = \lim_{h\to 0}\frac{a-b}{a^n-b^n}$$

$$= \lim_{a\to b}\frac{a-b}{a^n-b^n} = \lim_{a\to b}\frac{a-b}{(a-b)(\underbrace{a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1}}_{\text{n terms}})} =$$

$$= \lim_{a\to b}\frac{1}{a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1}} = \lim_{a\to b}\frac{1}{b^{n-1}+b^{n-2}b+\cdots+bb^{n-2}+b^{n-1}}$$

$$= \frac{1}{\underbrace{b^{n-1}+b^{n-1}+\cdots+b^{n-1}+b^{n-1}}_{\text{n terms}}} = \frac{1}{n\,b^{n-1}} = \frac{1}{n(\sqrt[n]{x})^{n-1}} = \frac{1}{nx^{\frac{n-1}{n}}} = \frac{1}{n}x^{\frac{1-n}{n}} = \frac{1}{n}x^{\frac{1}{n}-1} \quad \checkmark$$

<u>Note</u>: When dealing with root functions differentiation, always write them in the rational-exponent form , as the following example shows:

<u>**Example (section 3.4 exercise 15)**</u>: Find $f'(x)$ in simplified form if $f(x) = \frac{x^{\frac{1}{3}}}{1-\sqrt{x}}$

<u>Solution</u>:

$$f(x) = \frac{x^{\frac{1}{3}}}{1-x^{\frac{1}{2}}}$$

$$f'(x) = \frac{\left\{x^{\frac{1}{3}}\right\}'\left\{1-x^{\frac{1}{2}}\right\}-\left\{x^{\frac{1}{3}}\right\}\left\{1-x^{\frac{1}{2}}\right\}'}{\left(1-x^{\frac{1}{2}}\right)^2}$$

$$= \frac{\left\{\frac{1}{3}x^{-\frac{2}{3}}\right\}\left\{1-x^{\frac{1}{2}}\right\}-\left\{x^{\frac{1}{3}}\right\}\left\{-\frac{1}{2}x^{-\frac{1}{2}}\right\}}{\left(1-x^{\frac{1}{2}}\right)^2} = \frac{\frac{1}{3}x^{-\frac{2}{3}}-\frac{1}{3}x^{-\frac{2}{3}+\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{3}-\frac{1}{2}}}{\left(1-x^{\frac{1}{2}}\right)^2}$$

$$= \frac{\frac{1}{3}x^{-\frac{2}{3}}-\frac{1}{3}x^{-\frac{1}{6}}+\frac{1}{2}x^{-\frac{1}{6}}}{\left(1-x^{\frac{1}{2}}\right)^2} = \frac{\frac{1}{3}x^{-\frac{2}{3}}+\frac{1}{6}x^{-\frac{1}{6}}}{\left(1-x^{\frac{1}{2}}\right)^2} \quad \checkmark$$
