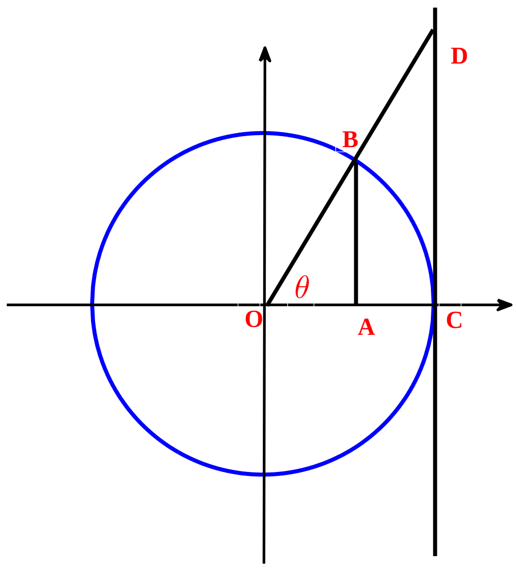
Derivatives of Trigonometric Functions
section 3.9

Consider the unit circle ; so we have $OC = 1$ in the picture below.

Because of similarity between the two triangles $\overset{\triangle}{OCD}$ and $\overset{\triangle}{OAB}$ we have

$$\frac{CD}{OC} = \frac{AB}{OA} \quad \Rightarrow \quad \frac{CD}{1} = \frac{\sin\theta}{\cos\theta} \quad \Rightarrow \quad CD = \tan\theta$$



Because of the equality $CD = \tan\theta$, the axis that passes through the points C and D is called the tangent axis.

Now assume for a moment that the radius of the circle is r. The sector OBC faces the polar angle θ.

angle	area of associated sector
2π	πr^2
θ	**?**

$$\Rightarrow \quad ? = \frac{(\theta)(\pi r^2)}{2\pi} = \frac{1}{2}\theta r^2$$

area of sector with polar angle $\theta = \frac{1}{2}\theta r^2$

Note: This formula is true only if θ is measured in radians.

Now let us assume for a moment that the above circle has radius 1. Then by putting $r = 1$ in the above formula we have:

$$\text{area of sector } OBC = \frac{1}{2}\theta$$

Now note that

$$\text{area of triangle } OAB < \text{area of sector } OBC < \text{area of triangle } OCD$$

$$\frac{1}{2}(OA)(AB) < \frac{1}{2}\theta < \frac{1}{2}(OC)(CD)$$

$$\frac{1}{2}(\cos\theta)(\sin\theta) < \frac{1}{2}\theta < \frac{1}{2}(1)(\tan\theta)$$

multiply by 2 and divide by $\sin\theta$

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Now let $\theta \to 0^+$ and use the Sandwich Theorem to get

$$\lim_{\theta\to 0^+} \frac{\theta}{\sin\theta} = 1$$

Then

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = \lim_{\theta\to 0^+} \frac{1}{\frac{\theta}{\sin\theta}} = \frac{1}{\lim_{\theta\to 0^+} \frac{\theta}{\sin\theta}} = \frac{1}{1} = 1 \qquad (*)$$

Now we calculate the left-hand limit:

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = \lim_{x\to 0^+} \frac{\sin(-x)}{-x} \qquad \textbf{change of variable } \mathbf{x} = -\theta$$

$$= \lim_{x\to 0^+} \frac{-\sin x}{-x} \qquad \textbf{using the identity } \mathbf{\sin(-x) = -\sin x}$$

$$= \lim_{x\to 0^+} \frac{\sin x}{x}$$

$$= 1 \qquad \textbf{using the identity (*)}$$

So ,

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1 \qquad (**)$$

From both $(*)$ and $(**)$ we have

$$\boxed{\lim_{\theta\to\mathbf{0}} \frac{\sin\theta}{\theta} = \mathbf{1}}$$

<u>**Example (section 3.9 exercise 33)**</u>. Calculate $\lim_{\theta\to\mathbf{0}} \frac{\tan\theta}{\theta}$

<u>**Solution**</u>.

$$\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = \lim_{\theta\to 0} \frac{\frac{\sin\theta}{\cos\theta}}{\theta} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \frac{1}{\cos\theta}$$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \lim_{\theta\to 0} \frac{1}{\cos\theta} = (1)(\frac{1}{1}) = 1$$

<u>**Example (section 3.9 exercise 36)**</u>. Calculate $\lim_{x\to\infty} \frac{\sin(\frac{2}{x})}{\sin(\frac{1}{x})}$

Solution. Using the change of variable $y = \frac{1}{x}$ we can write

$$\lim_{x\to\infty} \frac{\sin(\frac{2}{x})}{\sin(\frac{1}{x})} = \lim_{y\to 0} \frac{\sin(2y)}{\sin(y)} = \lim_{y\to 0} \frac{\frac{\sin(2y)}{y}}{\frac{\sin y}{y}} = \lim_{y\to 0} \frac{2\,\frac{\sin(2y)}{2y}}{\frac{\sin y}{y}} = \frac{(2)(1)}{1} = 2$$

We now calculate the derivative of the sine function: For this we use

$$\begin{aligned}
\sin'(x) &= \lim_{\Delta x\to 0} \tfrac{\sin(x+\Delta x)-\sin x}{\Delta x} \\
&= \lim_{\Delta x\to 0} \tfrac{2}{\Delta x}\left[\sin\left(\tfrac{\Delta x}{2}\right)\,\cos\left(x+\tfrac{\Delta x}{2}\right)\right] \qquad \text{using} \quad \sin a - \sin b = 2\sin(\tfrac{a-b}{2})\cos(\tfrac{a+b}{2}) \\
&= \lim_{\Delta x\to 0} \tfrac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)} \lim_{\Delta x\to 0} \cos\left(x+\tfrac{\Delta x}{2}\right) \\
&= (1)\cos(x+0) = \cos x
\end{aligned}$$

So

$$\boxed{\mathbf{\frac{d}{dx}\sin x = \cos x}}$$

Now we find the derivative of the cosine function:

$$\begin{aligned}
\tfrac{d}{dx}\cos x &= \tfrac{d}{dx}\sin\left(\tfrac{\pi}{2}-x\right) \\
&= \sin'\left(\tfrac{\pi}{2}-x\right)\left\{\tfrac{\pi}{2}-x\right\}' \\
&= \cos\left(\tfrac{\pi}{2}-x\right)\{-1\} \\
&= -\cos\left(\tfrac{\pi}{2}-x\right) \\
&= -\sin x
\end{aligned}$$

So

$$\boxed{\mathbf{\frac{d}{dx}\cos x = -\sin x}}$$

Now the derivative of the tangent function :

$$\begin{aligned}\tan' x &= \frac{d}{dx}\frac{\sin x}{\cos x} \\ &= \frac{\{\sin x\}'\{\cos x\} - \{\sin x\}\{\cos x\}'}{\cos^2 x} \\ &= \frac{\{\cos x\}\{\cos x\} - \{\sin x\}\{-\sin x\}}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

$$\boxed{\frac{\mathbf{d}}{\mathbf{dx}} \tan \mathbf{x} = \sec^{\mathbf{2}} \mathbf{x}}$$

Now the derivative of the cotangent function :

$$\begin{aligned}\cot' x &= \frac{d}{dx}\frac{\cos x}{\sin x} \\ &= \frac{\{\cos x\}'\{\sin x\} - \{\cos x\}\{\sin x\}'}{\sin^2 x} \\ &= \frac{\{-\sin x\}\{\sin x\} - \{\cos x\}\{\cos x\}}{\sin^2 x} \\ &= \frac{-(\cos^2 x + \sin^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x\end{aligned}$$

$$\boxed{\frac{\mathbf{d}}{\mathbf{dx}} \cot \mathbf{x} = -\csc^{\mathbf{2}} \mathbf{x}}$$

Now the derivative of the secant function :

$$\begin{aligned}
\sec' x &= \tfrac{d}{dx}\left(\tfrac{1}{\cos x}\right) \\
&= \tfrac{\{1\}'\{\cos x\}-\{1\}\{\cos x\}'}{\cos^2 x} \\
&= \tfrac{\{0\}\{\cos x\}-\{1\}\{-\sin x\}}{\cos^2 x} \\
&= \tfrac{\sin x}{\cos^2 x} = \tfrac{1}{\cos x}\tfrac{\sin x}{\cos x} = \sec x \tan x
\end{aligned}$$

$$\boxed{\mathbf{\tfrac{d}{dx}\sec x = \sec x \tan x}}$$

Now the derivative of the cosecant function :

$$\begin{aligned}
\csc' x &= \tfrac{d}{dx}\tfrac{1}{\sin x} \\
&= \tfrac{\{1\}'\{\sin x\}-\{1\}\{\sin x\}'}{\sin^2 x} \\
&= \tfrac{\{0\}\{\sin x\}-\{1\}\{\cos x\}}{\sin^2 x} \\
&= \tfrac{-\cos x}{\sin^2 x} = -\tfrac{1}{\sin x}\tfrac{\cos x}{\sin x} = -\csc x \cot x
\end{aligned}$$

$$\boxed{\mathbf{\tfrac{d}{dx}\csc x = -\csc x \cot x}}$$

Putting all these together:

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

Note: These formulae are true only if the unit of measurement is radian.

And , using the chain rule , we have the following formulae if interim variables are involved:

$$\frac{d}{dx}\sin u = (\cos u)\frac{du}{dx}$$

$$\frac{d}{dx}\cos u = (-\sin u)\frac{du}{dx}$$

$$\frac{d}{dx}\tan u = (\sec^2 u)\frac{du}{dx}$$

$$\frac{d}{dx}\cot u = (-\csc^2 u)\frac{du}{dx}$$

$$\frac{d}{dx}\sec u = (\sec u \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}\csc u = (-\csc u \cot u)\frac{du}{dx}$$

Example. Find the equation of the tangent line to the curve $y = \frac{\sec x}{1+\tan x}$ at the point where $x = 0$.

Solution.

$$y' = \frac{\{\sec x\}'\{1+\tan x\} - \{\sec x\}\{1+\tan x\}'}{(1+\tan x)^2}$$

$$= \frac{\{\sec x\ \tan x\}\{1+\tan x\} - \{\sec x\}\{\sec^2 x\}}{(1+\tan x)^2}$$

$$x = 0 \quad \Rightarrow \quad \text{slope} = \frac{(0)(1) - (1)(1)}{(1)^2} = -1$$

Also note that at $x = 0$ we have $y = 1$. Therefore:

$$\text{tangent line} \quad \Rightarrow \quad y - 1 = (-1)(x - 0) \quad \Rightarrow \quad y = -x + 1$$

Example (from the textbook). Find $\frac{dy}{dx}$ for $y = \tan^2(4x)$

Solution. Let $u = 4x$. Then

$$y = (\tan u)^2 \quad \Rightarrow \quad \frac{dy}{dx} = \left\{\frac{dy}{du}\right\}\left\{\frac{du}{dx}\right\}$$

$$= \left\{2(\tan u)(\sec^2 u)\right\}\left\{\frac{du}{dx}\right\} \qquad \textbf{power rule}$$

$$= \left\{2(\tan u)(\sec^2 u)\right\}\left\{4\right\}$$

$$= 8\ \tan(4x)\ \sec^2(4x) \qquad \checkmark$$

Example (from the textbook - pages 203 and 204). Find $\frac{dy}{dx}$ for $x^2 \tan y + y \sin x = 5$

Solution. This is an implicit differentiation. If the "prime notation" is meant to denote differentiation with respect to x , then we have

$$\Big\{\{x^2\}'\{\tan y\} + \{x^2\}\{\tan y\}'\Big\} + \Big\{\{y\}'\{\sin x\} + \{y\}\{\sin x\}'\Big\} = \{5\}'$$

$$\Big\{\{2x\}\{\tan y\} + \{x^2\}\{(\sec^2 y)\, y'\}\Big\} + \Big\{\{y'\}\{\sin x\} + \{y\}\{\cos x\}\Big\} = 0$$

factor out the term y′ $\quad \Rightarrow \quad (2x\ \tan y + y\ \cos x) + (x^2 \sec^2 y + \sin x)y' = 0$

$$\Rightarrow \quad y' = \frac{-(2x\ \tan y + y\ \cos x)}{(x^2 \sec^2 y + \sin x)} \qquad \checkmark$$

Example (section 3.9 exercise 30 - modified). Find $\dfrac{dy}{dx}$ for $y = \dfrac{1+\tan^3(3x^2-4)}{x^2 \sin(x^3)}$.

Solution. Set $u = 3x^2-4$ and $t = x^3$. If the "prime notation" is meant to denote differentiation with respect to x , then we have $y = \frac{1+(\tan(u))^3}{x^2 \sin t}$ so then:

$$\frac{dy}{dx} = \frac{\{1+(\tan u)^3\}'\{x^2 \sin t\} - \{1+(\tan u)^3\}\{x^2 \sin t\}'}{(x^2 \sin t)^2}$$

$$= \frac{\left\{3(\tan u)^2(\tan u)'\right\}\left\{x^2 \sin t\right\} - \left\{1+(\tan u)^3\right\}\left\{\{x^2\}'\{\sin t\}+\{x^2\}\{\sin t\}'\right\}}{x^4 \sin^2 t}$$

$$= \frac{\left\{3(\tan u)^2(\sec^2 u)(u')\right\}\left\{x^2 \sin t\right\} - \left\{1+(\tan u)^3\right\}\left\{\{2x\}\{\sin t\}+\{x^2\}\{(\cos t)(t')\}\right\}}{x^4 \sin^2 t}$$

$$= \frac{\left\{3(\tan u)^2(\sec^2 u)(6x)\right\}\left\{x^2 \sin t\right\} - \left\{1+(\tan u)^3\right\}\left\{\{2x\}\{\sin t\}+\{x^2\}\{(\cos t)(3x^2)\}\right\}}{x^4 \sin^2 t}$$

$$= \frac{\left\{18x^3 \tan^2 u \sec^2 u \sin t\right\} - \left\{1+\tan^3 u\right\}\left\{2x \sin t + 3x^4 \cos t\right\}}{x^4 \sin^2 t}$$

where $u = 3x^2 - 4$ and $t = x^3$