Derivatives of
Exponential and Logarithmic functions
(sections 3.11)

The Exponential functions are differentiable ; here are their derivatives:

$$\frac{d}{dx}\, e^x = e^x$$

$$\frac{d}{dx}\, a^x = a^x \ln(a)$$

Applying the chain rule , here are the formulas when interim variables are involved:

$$\frac{d}{dx}\, e^u = u'\, e^u$$

$$\frac{d}{dx}\, a^u = u'\, a^u \ln(a)$$

Example. Find $\frac{dy}{dx}$ if

(a) $y = e^{x^2-x}$

(b) $y = 3^{\sin x}$

Solution of part (a).

$$\frac{d}{dx}\left(e^{x^2-x}\right) = (x^2-x)'\, e^{x^2-x} = (2x-1)e^{x^2-x}$$

Solution of part (b).

$$\frac{d}{dx}\left(3^{\sin x}\right) = (\sin x)'\, 3^{\sin x}\ \ln(3) = (\cos x)\, 3^{\sin x}\ \ln(3)$$

Example. Find $\frac{dy}{dx}$ if

(a) $y = x^3\, e^{2x^2-1}$

(b) $y = \sqrt{x + e^{3x}}$

Solution of part (a). **Step 1**. Set $u = 2x^2 - 1$. Then

$$\left\{e^{2x^2-1}\right\}' = \frac{d}{dx}\left\{e^u\right\} = \left(\frac{d}{du}\, e^u\right)\left(\frac{du}{dx}\right) = e^u\, u' = \left(e^{2x^2-1}\right)(4x)$$

Step 2.

$$\frac{dy}{dx} = \{x^3\}'\{e^{2x^2-1}\}+\{x^3\}\{e^{2x^2-1}\}' = \{3x^2\}\left\{e^{2x^2-1}\right\}+\{x^3\}\left\{(4x)e^{2x^2-1}\right\}$$

$$= \{3x^2\}\{e^{2x^2-1}\} + (4x^4)e^{2x^2-1} = (3x^2+4x^4)e^{2x^2-1}$$

<u>Solution of part (b)</u>.

$$y = \left(x+e^{3x}\right)^{\frac{1}{2}} \quad \Rightarrow \quad y' = \tfrac{1}{2}\left(x+e^{3x}\right)^{-\frac{1}{2}}\left(x+e^{3x}\right)'$$

$$= \tfrac{1}{2}\left(x+e^{3x}\right)^{-\frac{1}{2}}\left(\{x\}'+\{3x\}'\,e^{3x}\right)$$

$$= \tfrac{1}{2}\left(x+e^{3x}\right)^{-\frac{1}{2}}\left(1+3\,e^{3x}\right)$$

$$= \tfrac{1+3e^{3x}}{2\sqrt{x+e^{3x}}}$$

The logarithmic functions are differentiable ; here are their derivatives:

$$\frac{\mathbf{d}}{\mathbf{dx}} \ln \mathbf{x} = \frac{\mathbf{1}}{\mathbf{x}} \qquad x > 0$$

$$\frac{\mathbf{d}}{\mathbf{dx}} \log_{\mathbf{a}} \mathbf{x} = \frac{\mathbf{1}}{\mathbf{x}\ln(\mathbf{a})}$$

Applying the chain rule , here are the formulas when interim variables are involved:

$$\frac{\mathbf{d}}{\mathbf{dx}} \ln \mathbf{u} = \frac{\mathbf{u}'}{\mathbf{u}} \qquad x > 0$$

$$\frac{\mathbf{d}}{\mathbf{dx}} \log_{\mathbf{a}} \mathbf{u} = \frac{\mathbf{u}'}{\mathbf{u}\ln(\mathbf{a})}$$

Example. Find $\frac{dy}{dx}$ if $y = \log_{0.7}(x^3 - x)$

Solution.

$$\frac{d\left(\log_{0.7}(x^3-x)\right)}{dx} = \frac{(x^3-x)'}{(x^3-x)\ln(0.7)} = \frac{3x^2-1}{(x^3-x)\ln(0.7)}$$