Drawing Graphs With Calculus
section 4.5

Example (section 4.5 - exercise 6). Draw the graph of the function $f(x) = \frac{x}{x^2+4}$

Solution.

Find the intervals on which the function is increasing and decreasing , and find relative extrema

$$f'(x) = \frac{(x)'(x^2+4)-(x)(x^2+4)'}{(x^2+4)^2} = \cdots = \frac{4-x^2}{(x^2+4)^2} = \frac{(2-x)(x+2)}{(x^2+4)^2}$$

$$f'(x) = 0 \quad \Rightarrow \quad x = \pm 2$$

		-2		2	
$x+2$	$-$	$\bullet$	$+$		$+$
$2-x$	$+$		$+$	$\bullet$	$-$
f'	$-$		$+$		$-$
f	$\searrow$		$\nearrow$		$\searrow$
		relative min		**relative max**	

Find the intervals on which the function is concave up and concave down , and find the inflection points

$$f''(x) = \frac{\{-x^2+4\}'\{(x^2+4)^2\}-\{-x^2+4\}\{(x^2+4)^2\}'}{(x^2+4)^4}$$

$$= \frac{\{-2x\}\{(x^2+4)^2\}-\{-x^2+4\}\{2(2x)(x^2+4)\}}{(x^2+4)^4}$$

$$= \frac{(-2x)\{x^2+4\}-\{-x^2+4\}\{4x\}}{(x^2+4)^3} \qquad \text{cancel out } (x^2+4)$$

$$= \frac{2x^3-24x}{(x^2+4)^3}$$

$$= \frac{2x(x^2-12)}{(x^2+4)^3} = \frac{2x(x-2\sqrt{3})(x+2\sqrt{3})}{(x^2+4)^3}$$

$$f''(x) = 0 \quad \Rightarrow \quad x = 0\,,\ \pm 2\sqrt{3}$$

		$-2\sqrt{3}$		0		$2\sqrt{3}$	
$x+2\sqrt{3}$	$-$	•	$+$		$+$		$+$
x	$-$		$-$	•	$+$		$+$
$x-2\sqrt{3}$	$-$		$-$		$-$	•	$+$
f''	$-$		$+$		$-$		$+$
f	$\cap$		$\cup$		$\cap$		$\cup$
		inflection point		**inflection point**		**inflection point**	

Find intercepts

$$x = 0 \quad \Rightarrow \quad y = 0$$

$$y = 0 \quad \Rightarrow \quad x = 0$$

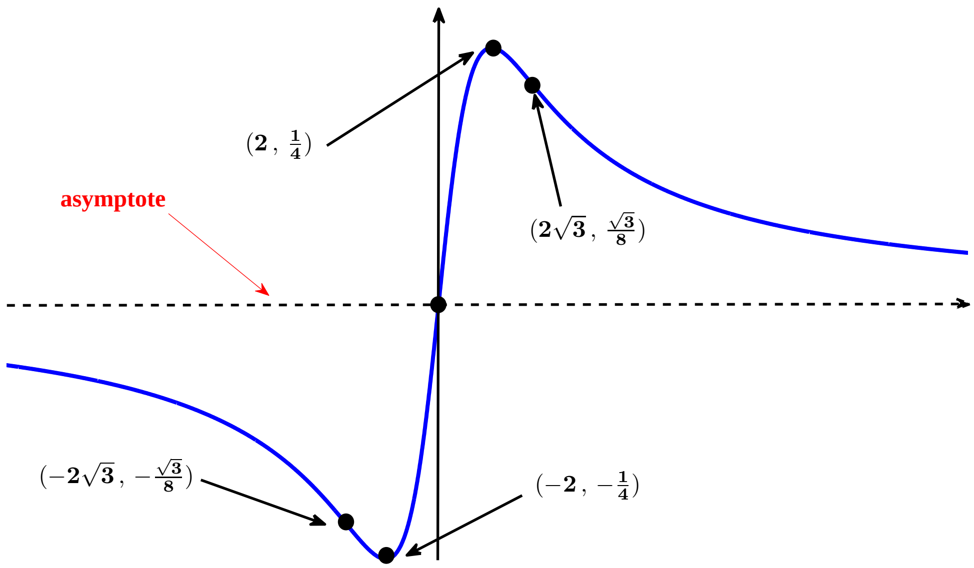
So, the only intercept is $(0, 0)$

Find asymptotes :

There is no vertical or any oblique asymptotes. But , there is horizontal asymptote which is found through this:

$$\lim_{x \to \pm\infty} \frac{x}{x^2+4} = 0$$

so , the only horizontal asymptote is the line $y = 0$



More examples will be added to this file

Example (section 4.5 - exercise 14). Draw the graph of the function $f(x) = \frac{x^3}{x^2-4}$

Solution.

Determine the domain if it is not indicated in the question :

The domain consists of all the points $x \neq \pm 2$

Find the intervals on which the function is increasing and decreasing , and find relative extrema

$$f'(x) = \frac{(x^3)'(x^2-4)-(x^3)(x^2-4)'}{(x^2-4)^2} = \frac{(3x^2)(x^2-4)-(x^3)(2x)}{(x^2-4)^2} = \frac{3x^4-12x^2-2x^4}{(x^2-4)^2} = \frac{x^4-12x^2}{(x^2-4)^2} = \frac{x^2(x^2-12)}{(x^2-4)^2}$$

$$f'(x) = 0 \quad \Rightarrow \quad x = 0 \ , \ \pm\sqrt{12}$$

Only the term $x^2 - 12$ needs to be checked for its sign.

		$-\sqrt{12}$		$\sqrt{12}$	
$x+\sqrt{12}$	$-$	•	$+$		$+$
$x-\sqrt{12}$	$-$		$-$	•	$+$
f'	$+$		$-$		$+$
f	$\nearrow$		$\searrow$		$\nearrow$
		relative max		**relative min**	

Find the intervals on which the function is concave up and concave down , and find the inflection points

$$f''(x) = \frac{\{x^4-12x^2\}'\{(x^2-4)^2\}-\{x^4-12x^2\}\{(x^2-4)^2\}'}{(x^2-4)^4}$$

$$= \frac{\{4x^3-24x\}\{(x^2-4)^2\}-\{x^4-12x^2\}\{2(2x)(x^2-4)\}}{(x^2-4)^4}$$

$$= \frac{(4x^3-24x)\{x^2-4\}-\{x^4-12x^2\}\{4x\}}{(x^2-4)^3} \qquad \text{cancel out } (x^2-4)$$

$$= \frac{8x^3+96x}{(x^2-4)^3}$$

$$= \frac{8x(x^2+12)}{(x^2-4)^3} = \frac{8x(x^2+12)}{(x-2)^3(x+2)^3}$$

$$f''(x) = 0 \quad \Rightarrow \quad x = 0$$

		-2		0		2	
$x+2$	$-$	•	$+$		$+$		$+$
x	$-$		$-$	•	$+$		$+$
$x-2$	$-$		$-$		$-$	•	$+$
f''	$-$		$+$		$-$		$+$
f	$\cap$		$\cup$		$\cap$		$\cup$
				inflection point			

Find intercepts

$x = 0 \quad \Rightarrow \quad y = 0$

$y = 0 \quad \Rightarrow \quad x = 0$

So, the only intercept is $(0, 0)$

Find asymptotes :

Since the values $x = \pm 2$ make the denominator equal to zero,

the lines $x = \pm 2$ are the vertical asymptotes.

Since $\lim_{x \to \pm\infty} \frac{x^3}{x^2-4} = \pm\infty$, there is no horizontal asymptote.

Since in the rational function $\frac{x^3}{x^2-4}$ the degree of the numerator is equal to one plus the degree of the denominator , there exists an oblique asymptote. To find it , we must do a division:

Divide the highest degree term of $p_0(x) = x^3$ by that of $q(x) = x^2 - 4$. We need $\boxed{x}$

multiply $q(x)$ by $x \quad \Rightarrow \quad x^3 - 4x$

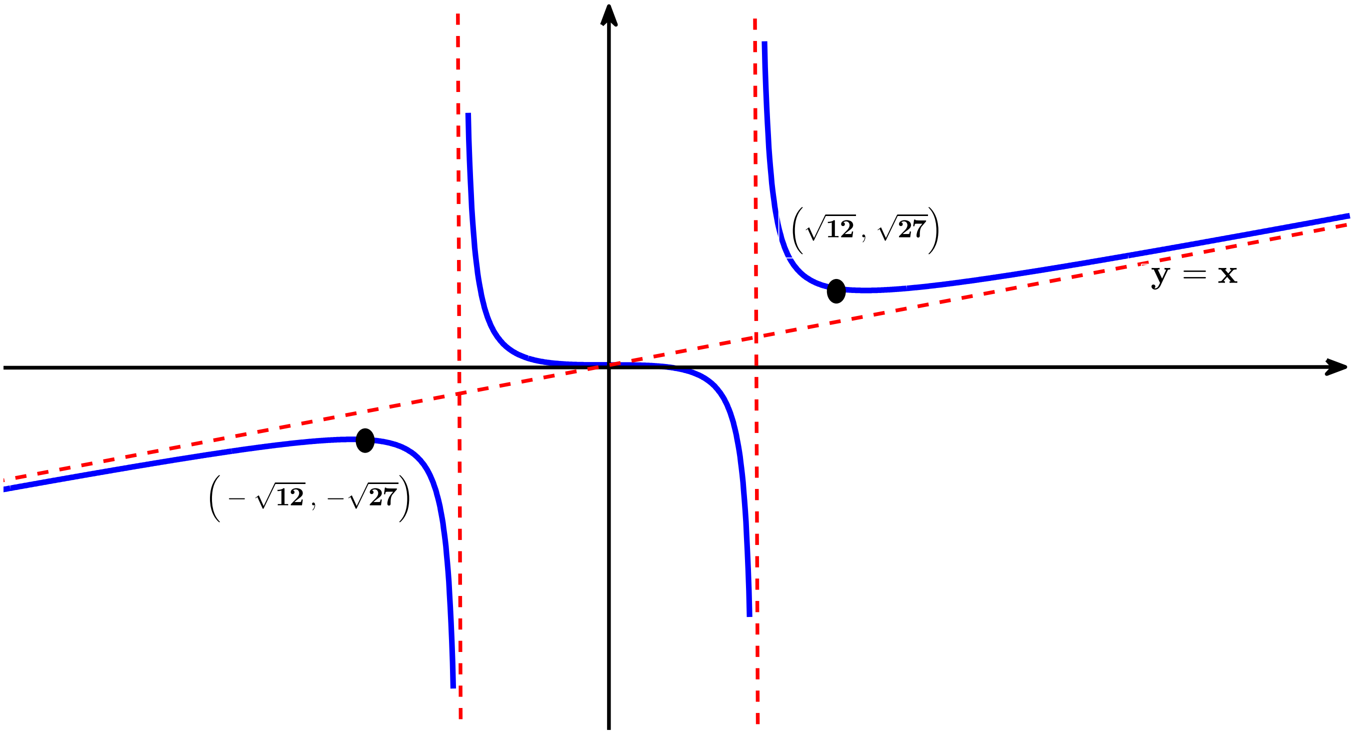
$$\overset{\text{change sign}}{\Rightarrow} \quad -x^3 + 4x \overset{\text{add to } p_0(x)}{\Rightarrow} \quad p_1(x) = 4x$$

We stop here because the degree of $p_1(x)$ is less than the degree of $p_0(x)$. Then

$$p_0(x) = x\,q(x) + 4x \quad \Rightarrow \quad x^3 = x(x^2-4) + 4x$$

$$\Rightarrow \quad \frac{x^3}{x^2-4} = x + \frac{4x}{x^2-4}$$

So , the line $y = x$ is the oblique asymptote.



Example (section 4.5 - exercise 16). Draw the graph of the function $f(x) = \frac{x^2+1}{x^2-1}$

Solution.

Determine the domain if it is not indicated in the question :

The domain consists of all the points $x \neq \pm 1$

Find the intervals on which the function is increasing and decreasing , and find relative extrema

$$f'(x) = \frac{(x^2+1)'(x^2-1)-(x^2+1)(x^2-1)'}{(x^2-1)^2} = \frac{(2x)(x^2+1)-(x^2+1)(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$$f'(x) = 0 \quad \Rightarrow \quad x = 0$$

		0	
$-4x$	$+$	$\bullet$	$-$
f'	$+$		$-$
f	$\nearrow$		$\searrow$
		relative max	

Find the intervals on which the function is concave up and concave down , and find the inflection points

$$
\begin{aligned}
f''(x) &= \frac{\{-4x\}'\{(x^2-1)^2\}-\{-4x\}\{(x^2-1)^2\}'}{(x^2-1)^4} \\
&= \frac{\{-4\}\{(x^2-1)^2\}-\{-4x\}\{2(2x)(x^2-1)\}}{(x^2-1)^4} \\
&= \frac{\{-4\}\{(x^2-1)\}-\{-4x\}\{2(2x)\}}{(x^2-1)^3} \qquad \text{cancel out } (x^2-4) \\
&= \frac{12x^2+4}{(x^2-1)^3} = \frac{4(3x^2+1)}{(x^2-1)^3} = \frac{4(3x^2+1)}{(x-1)^3(x+1)^3}
\end{aligned}
$$

$f''(x) = 0 \quad \Rightarrow \quad$ None

		-1		1	
$x+1$	$-$	$\bullet$	$+$		$+$
$x-1$	$-$		$-$	$\bullet$	$+$
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

Find intercepts

$x = 0 \quad \Rightarrow \quad y = -1$

$y = 0 \quad \Rightarrow \quad \text{None}$

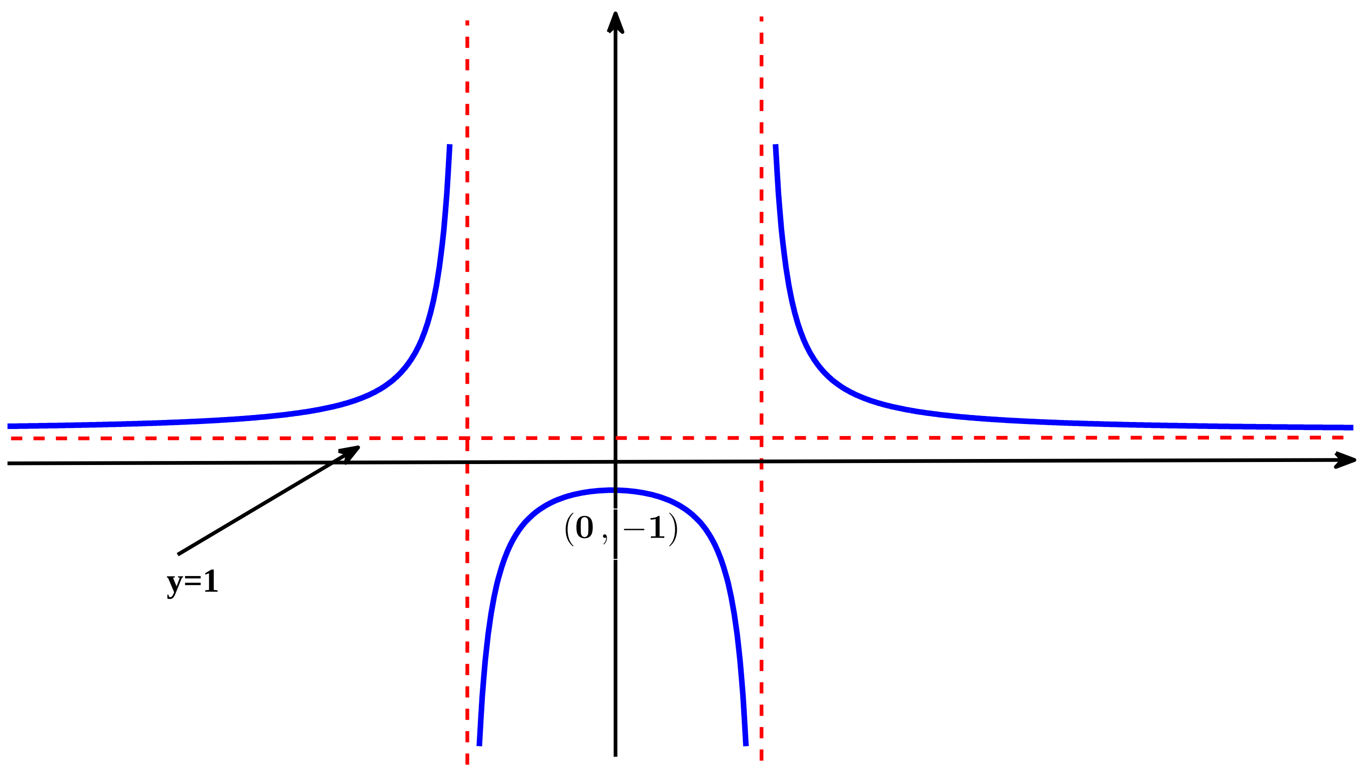
So, the only intercept is $(0, -1)$

Find asymptotes :

Since the values $x = \pm 1$ make the denominator equal to zero, the lines $x = \pm 1$ are the vertical asymptotes.

Since $\lim_{x \to \pm\infty} \frac{x^2+1}{x^2-1} = 1$, the line $y = 1$ is the only horizontal asymptote.

Since in the rational function $\frac{x^2+1}{x^2-1}$ the numerator and denominator are of equal degree , there exist no oblique asymptote.



(0, −1)
y=1