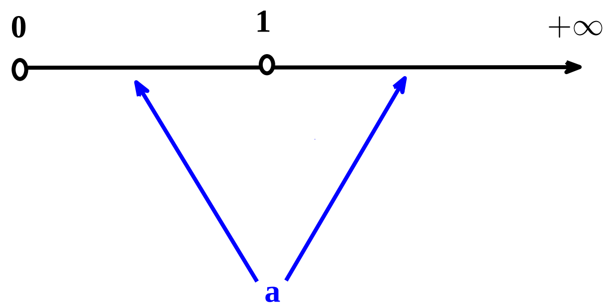
Introducing Exponential and Logarithmic functions

Definition. Functions $f : (-\infty\,,\,\infty) \overset{\text{onto}}{\to} (0,\infty)$ defined by

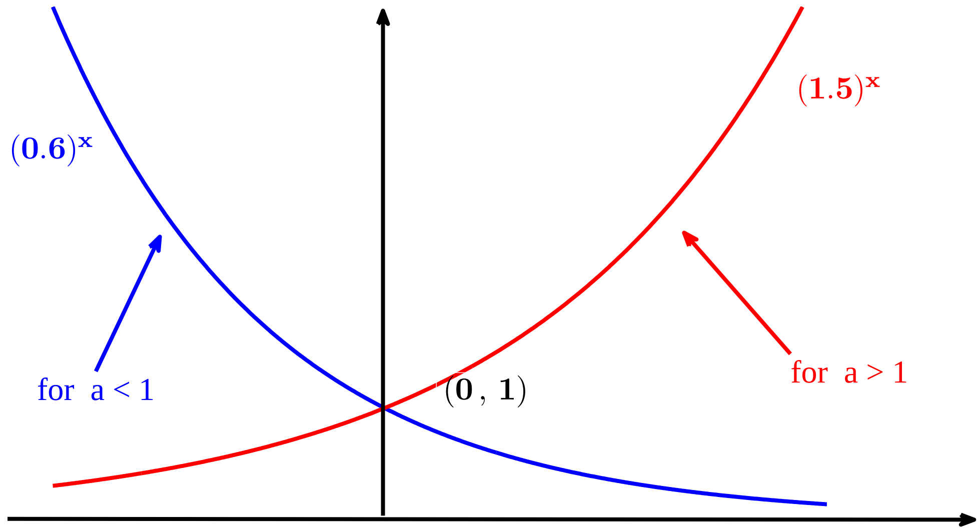
$$f(x) = a^x \qquad\qquad a > 0 \;\text{ and }\; a \neq 1$$

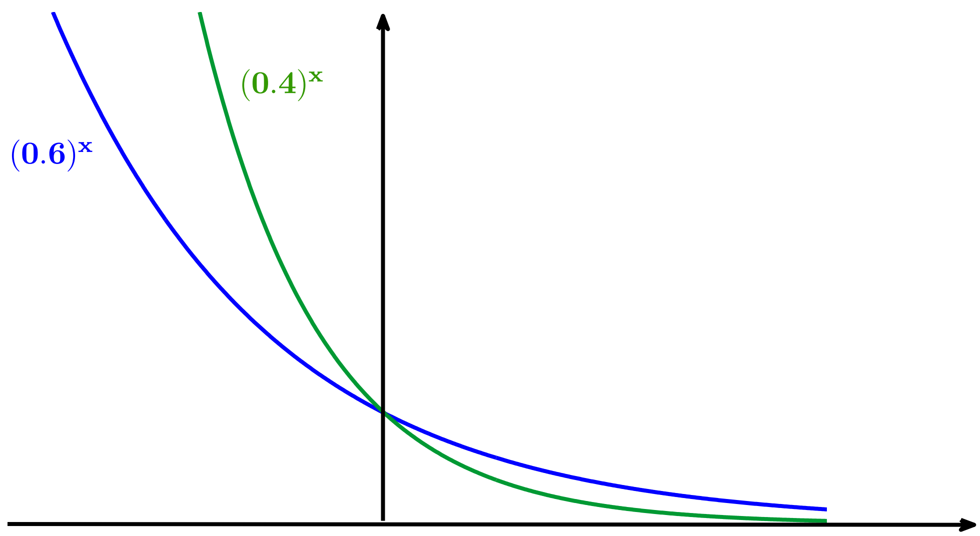
where the input variable is in the exponent, are called **exponential functions**.

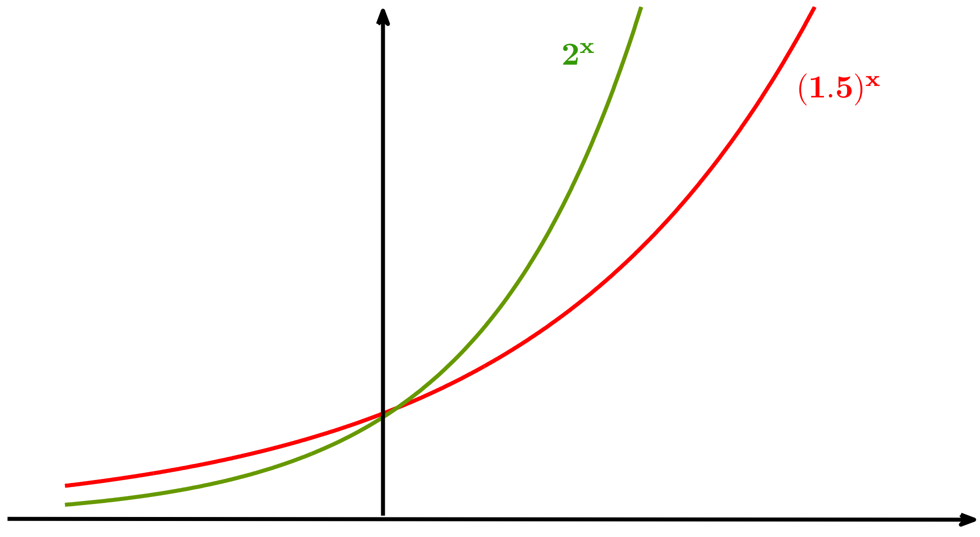


These functions have the common domain $(-\infty\,,\,\infty)$ and the common range $(0,\infty)$. Depending on a , their graphs is one of the following two : one is strictly increasing and the other is strictly decreasing. All pass through the point $(0,1)$, i.e.

$$a^0 = 1$$


(0.6)x
(1.5)x
for a < 1
(0 , 1)
for a > 1


(0.4)x
(0.6)x



If a is farther away from 1 the function enjoys more concavity

Theorem (algebraic properties of the exponential functions).

$$\mathbf{a^0 = 1}$$

$$\mathbf{a^{-x} = \frac{1}{a^x}}$$

$$\mathbf{a^{x+y} = a^x \cdot a^y}$$ **it changes sum to product**

$$\mathbf{a^{x-y} = \frac{a^x}{a^y}}$$ **it changes difference to division**

$$\mathbf{(a^x)^y = a^{xy}}$$

$$\mathbf{(ab)^x = a^x\, b^x}$$

$$\mathbf{\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}}$$

A frequently used base for the exponential functions is the Euler's Number e define by

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \qquad n \text{ being positive integer}$$

n	$\left(1+\frac{1}{n}\right)^n$
1	2.000000
3	2.370370
5	2.488320
10	2.593742
100	2.7048138
1000	2.704814
10000	2.718146
100000	2.718268
1000000	2.718280

Its value up to 10 decimal places is

$$e \approx 2.7182818284$$

With e as the base , we have :

$$\begin{cases} \mathbf{e^0} & = & \mathbf{1} \\ \mathbf{e^{-x}} & = & \frac{\mathbf{1}}{\mathbf{e^x}} \\ \mathbf{e^{x+y}} & = & \mathbf{e^x \cdot e^y} \\ \mathbf{e^{x-y}} & = & \frac{\mathbf{e^x}}{\mathbf{e^y}} \\ \mathbf{(e^x)^y} & = & \mathbf{e^{xy}} \end{cases}$$

As we have seen , the exponential functions are one-to-one (they are either strictly increasing or strictly decreasing). Therefore they enjoy having inverse functions. The inverse function to the exponential function with base a

$$x \mapsto a^x$$

is denoted by $\log_a$, and it is called the **logarithmic function** with base a.

$$\mathbf{y = a^x} \qquad \Longleftrightarrow \qquad \log_{\mathbf{a}}(\mathbf{y}) = \mathbf{x}$$

Example. Solve the equation $3^{2x-5} = 7$.

Solution.

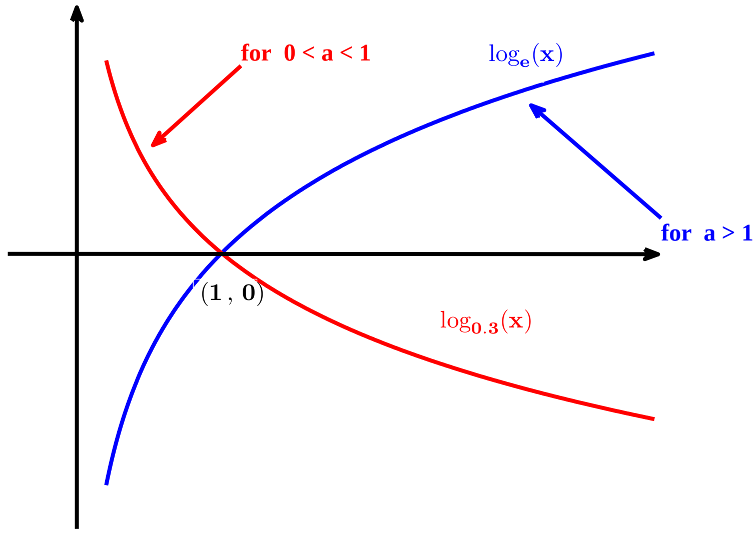
$$3^{2x-5} = 7 \quad \Rightarrow \quad 2x - 5 = \log_3(7) \quad \Rightarrow \quad x = \frac{5 + \log_3(7)}{2} \qquad \checkmark$$

Since the functions $\log_a(x)$ and a^x are inverses of each other , each undoes the action of the other one :

$$\boxed{\mathbf{a}^{\log_{\mathbf{a}}(\mathbf{x})} = \mathbf{x} \qquad \mathbf{x} > \mathbf{0}}$$

$$\boxed{\log_{\mathbf{a}}(\mathbf{a}^{\mathbf{x}}) = \mathbf{x} \qquad \text{for all } \mathbf{x}'s}$$

By taking the mirror images with respect to the line $y = x$ of the graphs of the exponential functions , we find the graphs of the logarithmic functions:


for 0 < a < 1
$\log_e(x)$
for a > 1
(1 , 0)
$\log_{0.3}(x)$

Theorem (algebraic properties of the logarithmic functions).

$$\begin{cases} \log_{\mathbf{a}}(\mathbf{1}) &= \mathbf{0} \\ \log_{\mathbf{a}}(\frac{\mathbf{1}}{\mathbf{x}}) &= -\log_{\mathbf{a}}(\mathbf{x}) \\ \log_{\mathbf{a}}(\mathbf{xy}) &= \log_{\mathbf{a}}(\mathbf{x}) + \log_{\mathbf{a}}(\mathbf{y}) \\ \log_{\mathbf{a}}(\frac{\mathbf{x}}{\mathbf{y}}) &= \log_{\mathbf{a}}(\mathbf{x}) - \log_{\mathbf{a}}(\mathbf{y}) \\ \log_{\mathbf{a}}(\mathbf{x}^{\mathbf{y}}) &= \mathbf{y}\ \log_{\mathbf{a}}(\mathbf{x}) \\ \log_{\mathbf{a}}(\mathbf{a}) &= \mathbf{1} \end{cases}$$

And , when the base is e :

$$\begin{cases} \ln(\mathbf{1}) &= \mathbf{0} \\ \ln(\frac{\mathbf{1}}{\mathbf{x}}) &= -\ln(\mathbf{x}) \\ \ln(\mathbf{xy}) &= \ln(\mathbf{x}) + \ln(\mathbf{y}) \\ \ln(\frac{\mathbf{x}}{\mathbf{y}}) &= \ln(\mathbf{x}) - \ln(\mathbf{y}) \\ \ln(\mathbf{x}^{\mathbf{y}}) &= \mathbf{y}\ \ln(\mathbf{x}) \\ \ln(\mathbf{e}) &= \mathbf{1} \end{cases}$$

All the log-functions and all the exponential functions are directly related:

$$\boxed{\log_{\mathbf{a}}(\mathbf{x}) = \frac{\ln \mathbf{x}}{\ln \mathbf{a}}} \qquad \boxed{\mathbf{a}^{\mathbf{x}} = \mathbf{e}^{\mathbf{x} \ln \mathbf{a}}}$$

Example.

$$\log_{10}(2) = \frac{\ln(2)}{\ln(10)} \qquad\qquad 5^{\sqrt{2}} = e^{\sqrt{2}\,\ln(5)}$$
