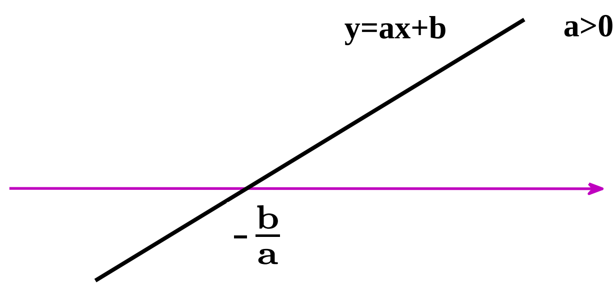
Increasing and Decreasing Functions
sections 4.2

Consider a line $y = ax + b$ with $a > 0$ (for example $y = 2x + 1$). The slope of this line is the number a which is assumed to be positive. The graph of it is

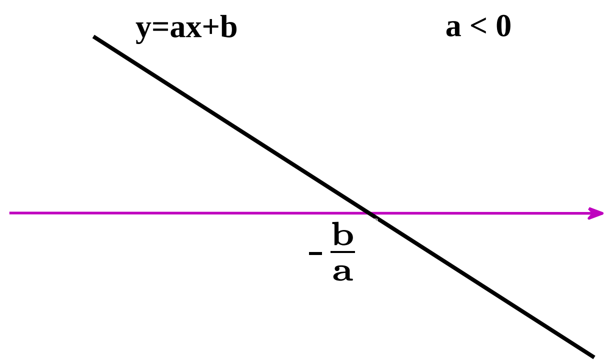


As one can see from the graph , for $x < -\frac{b}{a}$ the values of $y = ax+b$ are negative while for $x > -\frac{b}{x}$ they have positive values. Therefore we have the following table:

$-\frac{\mathbf{b}}{\mathbf{a}}$

y=ax+b	−	+

Now consider a line $y = ax + b$ with $a < 0$ (for example $y = -3x+7$). The slope of this line is the number a which is assumed to be negative. The graph of it is



As one can see from the graph , for $x < -\frac{b}{a}$ the values of $y = ax+b$ are positive while for $x > -\frac{b}{x}$ they have negative values. Therefore we have the following table:

So in both cases (whether a is positive or negative) , here is the rule:

		$-\frac{\mathbf{b}}{\mathbf{a}}$	
y=ax+b	$+$		$-$

		$-\frac{\mathbf{b}}{\mathbf{a}}$	
y=ax+b	**opposite to the sign of a**		**the same sign as a**

Definition. A function f is said to be **increasing** on an interval

I if

$$\left.\begin{array}{c}\text{for all } x_1\,,\ x_2 \in I \\ x_1 < x_2\end{array}\right\} \quad \Rightarrow \quad f(x_1) < f(x_2)$$

Definition. A function f is said to be **decreasing** on an interval I if

$$\left.\begin{array}{c}\text{for all } x_1\,,\ x_2 \in I \\ x_1 < x_2\end{array}\right\} \quad \Rightarrow \quad f(x_1) > f(x_2)$$

Theorem (Increasing-Decreasing Test).

(i) If for all x's in the interval I we have $f'(x) \geq 0$ with the equality , i.e. the case $f'(x) = 0$, happening at none or only a finite number of points of I, then f is increasing on I.

(ii) If for all x's in the interval I we have $f'(x) \leq 0$ with the equality , i.e. the case $f'(x) = 0$, happening at none or only a finite number of points of I, then f is decreasing on I.

Example (section 4.2 exercise 6). Determine the intervals on which the function $f(x) = 5 + 2x - 4x^2$ is increasing and increasing.

Solution.

$$f'(x) = 2 - 8x$$

$$2 - 8x = 0 \quad \Rightarrow \quad x = \frac{1}{4}$$

	$\frac{1}{4}$	
2−8x	+	−
f′	+	−
f	↗	↘

So the function is increasing on $(-\infty\,,\,\frac{1}{4}]$ and is decreasing on $[\frac{1}{4}\,,\,\infty)$.

Example (section 4.2 exercise 7). Determine the intervals on which the function $f(x) = 2x^3 - 18x^2 + 4x + 1$ is increasing and increasing.

Solution.

$$f'(x) = 6x^2 - 36x + 4 = 6(x^2 - 6x + \frac{4}{6}) = 6(x^2 - 6x + \frac{2}{3})$$

$$\text{(discriminant)} \quad b^2 - 4ac = 36 - (4)(1)(\frac{2}{3}) = 36 - \frac{8}{3} = \frac{108 - 8}{3} = \frac{100}{3}$$

$$x^2 - 6x + \frac{2}{3} = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{\frac{100}{3}}}{2} = \frac{6 \pm \frac{10}{\sqrt{3}}}{2} = 3 \pm \frac{5}{\sqrt{3}}$$

$$f'(x) = 6(x^2 - 6x + \frac{2}{3}) = 6\left(x - (3 + \frac{5}{\sqrt{3}})\right)\left(x - (3 - \frac{5}{\sqrt{3}})\right)$$

		$\mathbf{3-\frac{5}{\sqrt{3}}}$	$\mathbf{3+\frac{5}{\sqrt{3}}}$
$\mathbf{x-\left(3-\frac{5}{\sqrt{3}}\right)}$	$-$ $\bullet$	$+$	$+$
$\mathbf{x-\left(3+\frac{5}{\sqrt{3}}\right)}$	$-$	$-$ $\bullet$	$+$
$\mathbf{f'}$	$+$	$-$	$+$
$\mathbf{f}$	$\nearrow$	$\searrow$	$\nearrow$

So , the function f is increasing on the intervals $(-\infty\,,\,3-\frac{5}{\sqrt{3}}]$ and $[3+\frac{5}{\sqrt{3}}\,,\,\infty)$ and is decreasing on $[3-\frac{5}{\sqrt{3}}\,,\,3+\frac{5}{\sqrt{3}}]$.

Example (section 4.2 exercise 13). Determine the intervals on which the function $f(x)=x^4-4x^3-8x^2+48x+24$ is increasing and increasing.

Solution.

$$f'(x)=4x^3-12x^2-16x+48=4(x^3-3x^2-4x+12) \qquad (*)$$

Now here check whether any of the factors of 12 or their negatives is a root of $x^3-3x^2-4x+12$?. Indeed , the number 2 is a root , therefore by long division we get:

$$x^3 - 3x^2 - 4x + 12 = (x-2)(x^2 - x - 6) \qquad (*)$$

Now back to $(*)$ we continue:

$$f'(x) = 4(x-2)(x^2 - x - 6) \qquad (**)$$

Now we find the roots of $x^2 - x - 6$ to factorize it :

$$x^2 - x - 6 = 0 \quad \Rightarrow \quad x = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm \sqrt{25}}{2} = \frac{1 \pm 5}{2} = \begin{cases} 3 \\ -2 \end{cases}$$

Therefore , back to $(**)$ we will have :

$$f'(x) = 4(x-2)(x-3)(x+2)$$

		-2		2		3	
$x+2$	$-$	•	$+$		$+$		$+$
$x-2$	$-$		$-$	•	$+$		$+$
$x-3$	$-$		$-$		$-$	•	$+$
f'	$-$		$+$		$-$		$+$
f	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

Example (section 4.2 exercise 19). Determine the intervals on which the function $f(x) = \frac{x^3}{x+1}$ is increasing and decreasing.

Solution.

$$f'(x) = \frac{(x^3)'(x+1)-(x^3)(x+1)'}{(x+1)^2} = \frac{(3x^2)(x+1)-(x^3)(1)}{(x+1)^2}$$

$$= \frac{3x^3+3x^2-x^3}{(x+1)^2} = \frac{2x^3+3x^2}{(x+1)^2} = \frac{x^2(2x+3)}{(x+1)^2}$$

The only term whose sign must be determined is the term $2x+3$ because the other terms are positive (and therefore their signs is known).

		$-\frac{3}{2}$
2x+3	$-$	$+$
$\mathbf{f'}$	$-$	$+$
f	$\searrow$	$\nearrow$

So , the function is decreasing on the interval $(-\infty\,,\,-\frac{3}{2}]$ and is increasing on the interval $[-\frac{3}{2}\,,\,\infty)$. Note that the point $x=-1$ is not in the domain of the function , so in fact the graph of the function is formed of two pieces over the interval $[-\frac{3}{2}\,,\,\infty)$. However, the concept of "plotting graphs" is not the main subject here in section 4.2 , therefore we ignore these details in this section.

Example (section 4.2 exercise 22). Determine the intervals on which the function $f(x)=x^2\,e^{-x}$ is increasing and decreasing.

Solution.

$$
\begin{aligned}
f'(x) &= (x^2)'(e^{-x})+(x^2)(e^{-x})' \\
&= (2x)(e^{-x})+(x^2)(-e^{-x}) \\
&= (2x-x^2)e^{-x}=x(2-x)e^{-x}
\end{aligned}
$$

Since e^{-x} is a positive number $\Big($look at the graph of the exponential function ; alternatively we know that the exponential functions have the range $(0,\infty)$ and therefore they take on positive values$\Big)$, we only need to determine the sign of $x(2-x)$ in order to determine the sign of $f'(x)$.

		0		2	
x	$-$	•	$+$		$+$
$2-x$	$+$		$+$	•	$-$
f'	$-$		$+$		$-$
f	↘		↗		↘

The function f is decreasing on the intervals $(-\infty\,,\,0]$ and $[2\,,\,\infty)$ and it is increasing on the interval $[0\,,\,2]$