Inverse Functions

section 1.6

Definition. A function f is called **one-to-one** if arbitrary distinct values $s \neq t$ in the domain result in distinct values $f(s) \neq f(t)$. Graphically, every horizontal line must intersect the graph in at most one point. Symbolically:

$$s \neq t \qquad \Rightarrow \qquad f(s) \neq f(t)$$

equivalently:

$$f(s) = f(t) \qquad \Rightarrow \qquad s = t$$

Now take an arbitrary one-to-one function f and then find the mirror image of its graph with respect to the line $y = x$. Since the graph of f has the property that horizontal lines intersects the graph in at most one point, the mirror image has the property that the every vertical line intersects the graph in at most one point; therefore the mirror image is eligible to be the graph of a function. We denote this new function by f^{-1} and call it the **inverse function** of f.

$$(a,b) \text{ is on the graph of } f \qquad \Leftrightarrow \qquad (b,a) \text{ is on the graph of } f^{-1}$$

$$\text{domain of } f \quad = \quad \text{range of } f^{-1}$$

$$\text{range of } f \quad = \quad \text{domain of } f^{-1}$$

Note that

$$\begin{aligned} y = f(x) \qquad &\Leftrightarrow \qquad (x,y) \text{ is on the graph of } f \\ &\Leftrightarrow \qquad (y,x) \text{ is on the graph of } f^{-1} \\ &\Leftrightarrow \qquad x = f^{-1}(y) \end{aligned}$$

Then

$$y = f(x) = f(f^{-1}(y))$$

i.e.

$$\boxed{y = f(f^{-1}(y))}$$

Similarly one can verify that

$$\boxed{x = f^{-1}(f(x))}$$

So whatever one of the functions f or f^{-1} does, the other one undoes the action.

Fact. A function is invertible if and only if it is one-to-one

Example. Show that the function $f(x) = \frac{x-3}{2x+1}$ is one-to-one on its domain $D = \{x \; : \; x \neq -\frac{1}{2}\}$. Then find $f^{-1}(t)$ for all t.

Solution. Step 1. We must show that

$$f(s) = f(t) \qquad \Rightarrow \qquad s = t$$

Here is how:

$$f(s) = f(t) \qquad \Rightarrow \qquad \frac{s-3}{2s+1} = \frac{t-3}{2t+1} \qquad \Rightarrow \qquad (s-3)(2t+1) = (t-3)(2s+1) \qquad \Rightarrow$$

$$2st + s - 6t - 3 = 2st + t - 6s - 3 \qquad \Rightarrow \qquad s - 6t = t - 6s \qquad \Rightarrow \qquad s + 6s = t + 6t$$

$$\Rightarrow \qquad 7s = 7t \qquad \Rightarrow \qquad s = t \qquad \checkmark$$

Step 2. We want to find the rule for $f^{-1}(t)$. Call it y. We want to find y in terms of t. This is how it is done:

$$y = f^{-1}(t) \qquad \Rightarrow \qquad f(y) = t \qquad \Rightarrow \qquad \frac{y-3}{2y+1} = t \qquad \Rightarrow \qquad (2y+1)t = y - 3$$

$$\Rightarrow \qquad 2yt + t = y - 3 \qquad \Rightarrow \qquad y(2t-1) = -t - 3 \qquad \Rightarrow \qquad y = \frac{-t-3}{2t-1}$$

$$\Rightarrow \qquad \boxed{f^{-1}(t) = \tfrac{-t-3}{2t-1}}$$

Question: How do we algebraically find the inverse function? See the following example:

Example. Find the inverse function of the function $f(x) = \frac{x+1}{x-2}$.

Solution. Take an arbitrary input variable x for f^{-1} and set $y = f^{-1}(x)$. Then

$$y = f^{-1}(x) \quad \Rightarrow \quad f(y) = x \quad \Rightarrow \quad \frac{y+1}{y-2} = x$$

$$\Rightarrow \quad x(y-2) = y+1 \quad \Rightarrow \quad xy - 2x - y - 1 = 0$$

$$\Rightarrow \quad y(x-1) - 2x - 1 = 0 \quad \Rightarrow \quad y = \frac{2x+1}{x-1}$$

$$\Rightarrow \quad \boxed{f^{-1}(x) = \tfrac{2x+1}{x-1}}$$

Definition An upward sloping function is called **strictly increasing** . Mathematically (it preserves ordering):

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$$

Definition An upward sloping function is called **strictly decreasing** . Mathematically (it reverses ordering):

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) > f(x_2)$$

Definition A function that is strictly increasing or strictly decreasing is called **strictly monotonic**.

Theorem A strictly monotonic function is one-to-one.