Lab 1 solutions

Section 1.5 exercise 21:

$$f_e(x) = \frac{f(x)+f(-x)}{2} = \frac{|x|+|-x|}{2} = \frac{2|x|}{2} = |x| \qquad \text{as } |-x| = |x|$$

$$f_o(x) = \frac{f(x)-f(-x)}{2} = \frac{|x|-|-x|}{2} = \frac{0}{2} = 0$$

Section 1.5 exercise 22

$$f_e(x) = \frac{f(x)+f(-x)}{2} = \frac{1}{2}\left\{\frac{x^3}{x^2+3} + \frac{(-x)^3}{(-x)^2+3}\right\} = \frac{1}{2}\left\{\frac{x^3}{x^2+3} + \frac{-x^3}{x^2+3}\right\} = 0$$

$$f_e(x) = \frac{f(x)-f(-x)}{2} = \frac{1}{2}\left\{\frac{x^3}{x^2+3} - \frac{(-x)^3}{(-x)^2+3}\right\} = \frac{1}{2}\left\{\frac{x^3}{x^2+3} - \frac{-x^3}{x^2+3}\right\} = \frac{1}{2}\left\{2\frac{x^3}{x^2+3}\right\} = \frac{x^3}{x^2+3}$$

Section 2.1 exercise 30

$$\lim_{x\to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x} = \lim_{x\to 0} \left\{\frac{\sqrt{1-x}-\sqrt{1+x}}{x}\right\}\left\{\frac{\sqrt{1-x}+\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}\right\}$$

$$= \lim_{x\to 0} \frac{(1-x)-(1+x)}{x(\sqrt{1-x}+\sqrt{1+x})} = \lim_{x\to 0} \frac{-2x}{x(\sqrt{1-x}+\sqrt{1+x})} = \lim_{x\to 0} \frac{-2}{\sqrt{1-x}+\sqrt{1+x}} = \frac{-2}{\sqrt{1-0}+\sqrt{1+0}} = 1$$

Section 2.1 exercise 31 The following table shows that the function $x^2-25 = (x+5)(x-5)$ takes on the negative values as $x \to 5^-$. Therefore:

$$\lim_{x\to 5^-} \frac{|x^2-25|}{x^2-25} = \lim_{x\to 5^-} \frac{-(x^2-25)}{x^2-25} = -1$$

		-5		5	
x+5	$-$	0	$+$		$+$
x−5	$-$		$-$	0	$+$
(x+5)(x−5)	$+$		$-$		$+$

Section 2.1 exercise 37

$$\lim_{x\to(-2)^+}\frac{\sqrt{x+3}-\sqrt{-x-1}}{\sqrt{x+2}}=\lim_{x\to(-2)^+}\left\{\frac{\sqrt{x+3}-\sqrt{-x-1}}{\sqrt{x+2}}\right\}\left\{\frac{\sqrt{x+3}+\sqrt{-x-1}}{\sqrt{x+3}+\sqrt{-x-1}}\right\}$$

$$=\lim_{x\to(-2)^+}\frac{(x+3)-(-x-1)}{\sqrt{x+2}(\sqrt{x+3}+\sqrt{-x-1})}=\lim_{x\to(-2)^+}\frac{2(x+2)}{\sqrt{x+2}(\sqrt{x+3}+\sqrt{-x-1})}$$

$$=\lim_{x\to(-2)^+}\frac{2\sqrt{x+2}}{\sqrt{x+3}+\sqrt{-x-1}}=\frac{2\ \textbf{times}\ 0}{1+1}=0$$

Section 2.1 exercise 39

$$\lim_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{2+x}-\sqrt{2-x}}=\lim_{x\to0}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{2+x}-\sqrt{2-x}}\right\}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\}\left\{\frac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}+\sqrt{2-x}}\right\}$$

$$=\lim_{x\to0}\left\{\frac{(1+x)-(1-x)}{(2+x)-(2-x)}\right\}\left\{\frac{1}{\sqrt{1+x}+\sqrt{1-x}}\right\}\left\{\frac{\sqrt{2+x}+\sqrt{2-x}}{1}\right\}$$

$$=\lim_{x\to0}\left\{\frac{2x}{2x}\right\}\left\{\frac{1}{\sqrt{1+x}+\sqrt{1-x}}\right\}\left\{\frac{\sqrt{2+x}+\sqrt{2-x}}{1}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{1}{\sqrt{1+x}+\sqrt{1-x}}\right\}\left\{\frac{\sqrt{2+x}+\sqrt{2-x}}{1}\right\} = \frac{2\sqrt{2}}{2\sqrt{2}} = 1$$

Question 3 Describe the domain of the function $y = \frac{\sqrt{x}+1}{1-\sqrt{1-x^2}}$

Solution **Step 1**: The denominator must be non-zero. But

$$1-\sqrt{1-x^2}=0 \quad \Longleftrightarrow \quad \sqrt{1-x^2}=1 \quad \Longleftrightarrow \quad -x^2=0 \quad \Longleftrightarrow \quad x=0$$

Therefore as the first restriction we must have $x \neq 0$.

Step 2: For $\sqrt{1-x^2}$ to make sense , we must have $1-x^2 \geq 0$. This is equivalent to

$$1-x^2 \geq 0 \quad \Longleftrightarrow \quad 1 \geq x^2 \quad \Longleftrightarrow \quad \sqrt{1} \geq \sqrt{x^2} \quad \Longleftrightarrow \quad 1 \geq |x| \quad \Longleftrightarrow \quad 1 \geq x \geq -1$$

Step 3: For $\sqrt{x}$ to make sense , we must have $x \geq 0$.

The intersection of these three restrictions is $0 < x \leq 1$. This is the domain of f.

$$D_f = (0\,,\,1]$$

Question 4 Plot the graph of the function $y = x + |x^2-4x+3|$

Solution

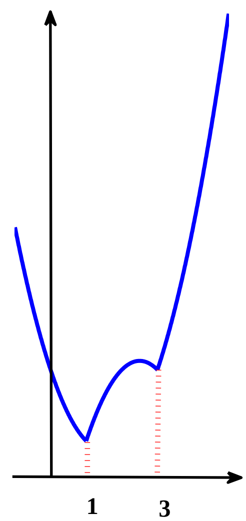
$$x^2-4x+3=0 \quad \Longleftrightarrow \quad x=1\ ,\ 3$$

For the sign of quadratic forms , it is true that between the two roots the sign is opposite sign to the coefficient of x^2 term. so , we have

$$y = \begin{cases} x+(x^2-4x+3) & x<1 \\ x-(x^2-4x+3) & 1 \leq x \leq 3 \\ x+(x^2-4x+3) & x>3 \end{cases}$$

$$\Rightarrow \quad y = \begin{cases} x^2-3x+3 & x<1 \\ -x^2+5x-3 & 1 \leq x \leq 3 \\ x^2-3x+3 & x>3 \end{cases}$$

		1		3	
X^2-4x+3	$+$		$-$		$+$



By plotting the graphs of these three quadratic forms over the intervals $(-\infty\,,\,1)$, $[1\,,\,3]$, and $(3\,,\,\infty)$ we will have:

Question 5 (challenging) Find the value(s) of k such that the function

$$f(x) = \begin{cases} \frac{\sqrt{1+kx}-1}{x} & x < 0 \\ (x-k)^2 + \frac{k}{2} - 1 & x > 0 \end{cases}$$

has a limit at the point $x = 0$.

Solution For the function f to have a limit at $x = 0$ we must have $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$.

But:

$$\begin{aligned}
\lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} \tfrac{\sqrt{1+kx}-1}{x} \\
&= \lim_{x \to 0^-} \tfrac{\sqrt{1+kx}-1}{x} \tfrac{\sqrt{1+kx}+1}{\sqrt{1+kx}+1} \\
&= \lim_{x \to 0^-} \tfrac{(\sqrt{1+kx})^2-(1)^2}{x(\sqrt{1+kx}+1)} \\
&= \lim_{x \to 0^-} \tfrac{(1+kx)-1}{x(\sqrt{1+kx}+1)} \\
&= \lim_{x \to 0^-} \tfrac{kx}{x(\sqrt{1+kx}+1)} \\
&= \lim_{x \to 0^-} \tfrac{k}{\sqrt{1+kx}+1} \\
&= \tfrac{k}{\sqrt{1+0}+1} = \tfrac{k}{2}
\end{aligned}$$

And:

$$\begin{aligned}
\lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} (x-k)^2 + \tfrac{k}{2} - 1 \\
&= (0-k)^2 + \tfrac{k}{2} - 1 = k^2 + \tfrac{k}{2} - 1
\end{aligned}$$

Then for f to have a limit at $x = 0$ we must equivalently have:

$$\lim_{\mathbf{x}\to\mathbf{0}^-} \mathbf{f(x)} = \lim_{\mathbf{x}\to\mathbf{0}^+} \mathbf{f(x)} \quad \Rightarrow \quad \frac{\mathbf{k}}{\mathbf{2}} = \mathbf{k^2} + \frac{\mathbf{k}}{\mathbf{2}} - \mathbf{1} \quad \Rightarrow \quad \mathbf{k^2 = 1} \quad \Rightarrow \quad \boxed{\mathbf{k = \pm 1}}$$

<u>Question 6</u>: Find the limits

$$\lim_{x\to-2} \frac{\sqrt[3]{x-6}+2}{x^3+8}$$

by finding factors of $(x+2)$ in both the numerator and denominator.

<u>Solution</u>: **<u>For the numerator</u>**: We use the identity $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ to write

$$\mathbf{numerator} = \sqrt[3]{x-6} + 2 = \frac{(\sqrt[3]{x-6}+2)\left\{(\sqrt[3]{x-6})^2 - (\sqrt[3]{x-6})(2) + (2)^2\right\}}{(\sqrt[3]{x-6})^2 - (\sqrt[3]{x-6})(2) + (2)^2}$$

$$= \frac{(\sqrt[3]{x-6})^3 + (2)^3}{\sqrt[3]{(x-6)^2} - (\sqrt[3]{x-6})(2) + (2)^2} = \frac{(x-6)+8}{(\sqrt[3]{x-6})^2 - (\sqrt[3]{x-6})(2) + (2)^2}$$

$$= \frac{x+2}{(\sqrt[3]{x-6})^2 - (\sqrt[3]{x-6})(2) + (2)^2}$$

<u>For the denominator</u>:

$$\mathbf{denominator} = x^3 + 8 = (x+2)(x^2 - 2x + 4)$$

Then by dividing the two expressions:

$$\frac{\mathbf{numerator}}{\mathbf{denominator}} = \frac{\frac{x+2}{\sqrt[3]{(x-6)^2} - (\sqrt[3]{x-6})(2) + (2)^2}}{(x+2)(x^2 - 4x + 4)} = \frac{1}{(x^2 - 2x + 4)\left((\sqrt[3]{x-6})^2 - (\sqrt[3]{x-6})(2) + (2)^2\right)}$$

Now by applying the limit $\lim_{x\to-2}$ we will have:

$$\lim_{x\to-2} \frac{\sqrt[3]{x-6}+2}{x^3+8} = \lim_{x\to-2} \frac{1}{(x^2 - 2x + 4)\left((\sqrt[3]{x-6})^2 - (\sqrt[3]{x-6})(2) + (2)^2\right)}$$

$$= \frac{1}{(4+4+4)(4-(-2)(2)+4)} = \frac{1}{144}$$

Question 7: Consider the function

$$y = \frac{1+\sqrt{x}}{1-\sqrt{x}} \qquad\qquad 0 \le x < 1$$

Find $f^{-1}(x)$. What is the domain of f^{-1} ?.

Solution: Let $z = f^{-1}(x)$. We need to find z in terms of x. Here is how:

$$z = f^{-1}(x) \quad \Rightarrow \quad f(z) = x \quad \Rightarrow \quad \frac{1+\sqrt{z}}{1-\sqrt{z}} = x \quad \Rightarrow \quad 1+\sqrt{z} = x(1-\sqrt{z})$$

$$\Rightarrow \quad 1+\sqrt{z} = x - x\sqrt{z} \quad \Rightarrow \quad \sqrt{z}(x+1) = x-1 \quad \Rightarrow \quad \sqrt{z} = \frac{x-1}{x+1}$$

$$\Rightarrow \quad z = \left(\frac{x-1}{x+1}\right)^2 \quad \Rightarrow \quad \boxed{f^{-1}(x) = \left(\tfrac{x-1}{x+1}\right)^2}$$

We remove the second part of the question which asks about the domain of f^{-1} , because it should be done after chapter 4 is studied.

Question 8: Consider the function

$$y = \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} \qquad\qquad -1 \le x < 0$$

Find $f^{-1}(x)$. What is the domain of f^{-1} ?.

Solution: Let $z = f^{-1}(x)$. We need to find z in terms of x. Here is how:

$$z = f^{-1}(x) \quad \Rightarrow \quad f(z) = x \quad \Rightarrow \quad \frac{1+\sqrt{1-z^2}}{1-\sqrt{1-z^2}} = x \quad \Rightarrow \quad 1+\sqrt{1-z^2} = x(1-\sqrt{1-z^2})$$

$$\Rightarrow \quad 1+\sqrt{1-z^2} = x - x\sqrt{1-z^2} \quad \Rightarrow \quad \sqrt{1-z^2}(x+1) = x-1 \quad \Rightarrow \quad \sqrt{1-z^2} = \frac{x-1}{x+1}$$

$$\Rightarrow \quad 1-z^2 = \left(\frac{x-1}{x+1}\right)^2 \quad \Rightarrow \quad z^2 = 1-\left(\frac{x-1}{x+1}\right)^2 \quad \Rightarrow \quad z = \pm\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}$$

Now we must decide on which sign $\pm$ to take. Well , from $z = f^{-1}(x)$ we know that z is in the range of f^{-1} and therefore is in the domain of f , so it must satisfy $-1 < z < 0$. Therefore the negative sign must be taken:

$$z = -\sqrt{1 - \left(\frac{x-1}{x+1}\right)^2} \quad \Rightarrow \quad \boxed{f^{-1}(x) = -\sqrt{1 - \left(\tfrac{x-1}{x+1}\right)^2}}$$

We remove the second part of the question which asks about the domain of f^{-1} , because it should be done after chapter 4 is studied.