Tutorial Session 3

1. Find the following limits using the Sandwich Theorem or Zero-time-bounded theorem whichever easier to work with:

(i) $\lim_{x\to 0} x^4\left(3+\cos\left(\frac{2}{x}\right)\right)$

$\boxed{\text{zero} \times \text{bounded} = \text{zero}}$

we know if $g(x) = 3+\cos\left(\frac{2}{x}\right)$ then we have

$-1 \leqslant \cos\left(\frac{2}{x}\right) \leqslant 1 \Rightarrow 2 \leqslant 3+\cos\left(\frac{2}{x}\right) \leqslant 4 \Rightarrow \left|3+\cos\left(\frac{2}{x}\right)\right| \leqslant 4$

So $g(x)$ is bounded, and if $f(x) = x^4$ then

$\lim_{x\to 0} x^4 = 0$, So by using the zero-times-bounded theorem we have

$$\lim_{x\to 0} x^4\left(3+\cos\left(\frac{2}{x}\right)\right) = 0$$

(ii) $\lim_{x\to\infty} \dfrac{5x^3 + x^2 \sin x \cos(x-1)}{2x^3+1}$

$$= \lim \frac{x^3\left\{5+\frac{1}{x}\sin x \cos(x-1)\right\}}{x^3\left(2+\frac{1}{x^3}\right)}$$

$$= \lim_{x\to\infty} \frac{5+\frac{1}{x}\sin x\cos(x-1)}{2+\frac{1}{x^3}} = \frac{5+0}{2+0} = \frac{5}{2}$$

Zero-times-bounded

(iii) $$\lim_{x\to\infty} \frac{x^2 \sin x + \cos^2\left(\frac{x}{2}\right)}{x^3+1}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^3+1} \sin x + \lim \frac{1}{x^3+1} \cos^2\left(\frac{x}{2}\right)$$

zero bounded zero bounded

$$= 0 + 0 = 0$$

iv) $$\lim_{x\to\infty} \frac{x^2 + 2\sin x}{2x^2 - \cos x}$$

$$= \lim_{x\to\infty} \frac{x^2\left(1 + \frac{2}{x^2}\sin x\right)}{x^2\left(2 - \frac{1}{x^2}\cos x\right)}$$

Zero-times-bounded

$$= \lim_{x\to\infty} \frac{1 + \frac{2}{x^2}\sin x}{2 - \frac{1}{x^2}\cos x}$$

$$= \frac{1+0}{2-0} = \frac{1}{2}$$

~~Exercise~~

Section 2.3 question 44

$$\lim_{x\to\infty} \frac{x+3}{2x-5} = \lim \frac{x(1+\frac{3}{x})}{x(2-\frac{5}{x})} = \lim_{x\to\infty} \frac{1+\frac{3}{x}}{2-\frac{5}{x}} = \frac{1}{2}$$

$\Rightarrow$ horizontal asymptote : $\boxed{y=\frac{1}{2}}$

To find ~~the~~ a vertical asymptote just let the denominator be equal to zero to get $x=\frac{5}{2}$, and then note that:

$$\lim_{x\to(\frac{5}{2})^+} \frac{x+3}{2x-5} = \frac{\frac{5}{2}+3}{0^+} = +\infty$$

$\Rightarrow$ the line $x=\frac{5}{2}$ is the vertical asymptote

$$\lim_{x\to\pm\infty} \frac{3x-1}{\sqrt{5+2x^2}} = \lim \frac{x\left(3-\frac{1}{x}\right)}{|x|\sqrt{\frac{5}{x^2}+2}}$$

$$= \lim_{x\to\pm\infty} \frac{x\left(3-\frac{1}{x}\right)}{\pm x\sqrt{\frac{5}{x^2}+2}}$$

$$= \lim_{x\to\pm\infty} \frac{3-\frac{1}{x}}{\pm\sqrt{\frac{5}{x^2}+2}} = \frac{3}{\pm\sqrt{2}}$$

$\Rightarrow$ the line $\boxed{y=\frac{3}{\pm\sqrt{2}}}$ are horizontal asymptotes.

Since no x can make the denominator zero, there is no vertical asymptotes.

Section 2.3 question 46

$$\lim_{x\to\pm\infty} \frac{\sqrt{5x^2+7}}{2x+3} = \lim_{x\to\pm\infty} \frac{|x|\sqrt{5+\frac{7}{x^2}}}{x\left(2+\frac{3}{x}\right)}$$

$$= \lim_{x\to\pm\infty} \frac{\pm x\sqrt{5+\frac{7}{x^2}}}{x\left(2+\frac{3}{x}\right)}$$

$$= \lim_{x\to\pm\infty} \frac{\pm\sqrt{5+\frac{7}{x^2}}}{2+\frac{3}{x}} = \frac{\pm\sqrt{5}}{2}$$

$\Rightarrow$ the lines $y = \pm\frac{\sqrt{5}}{2}$ are the horizontal asymptotes

To find vertical asymptotes one needs to set the denominator equal to zero. This gives $x = \frac{-3}{2}$. Now one can easily check that

lim

$$\lim_{x\to(-\frac{3}{2})^+} \frac{\sqrt{5x^2+7}}{2x+3} = \frac{\sqrt{\frac{73}{4}}}{0^+} = +\infty$$

$\Rightarrow$ the line $x=-\frac{3}{2}$ is the vertical asymptote.

Section 2.3 question 48

$$\lim_{x\to\pm\infty} \frac{3x^3+2x-1}{1-3x+x^2} = \lim_{x\to\pm\infty} \frac{x^3\left(3+\frac{2}{x^2}-\frac{1}{x^3}\right)}{x^2\left(\frac{1}{x^2}-\frac{3}{x}+1\right)}$$

$$= \lim_{x\to\pm\infty} x \cdot \frac{3+\frac{2}{x^2}-\frac{1}{x^3}}{\frac{1}{x^2}-\frac{3}{x}+1} = (\pm\infty)\cdot\left(\frac{3}{1}\right)$$

$$= \pm\infty$$

$\Rightarrow$ there is no horizontal asymptote.

To find the vertical asymptotes, set

the denominator equal to zero to get $x = \frac{3 \pm \sqrt{5}}{2}$. Since the numerator does not become zero at these two points, the lines $x = \frac{3 \pm \sqrt{5}}{2}$ are the vertical asymptotes

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Solution to question 3 of the lab Session 3

Part (i)

$$(g \circ f)(x) = g(f(x)) = g(x^2+1) = (x^2+1)^2 + (x^2+1) - 1$$

$$= (x^4 + 2x^2 + 1) + (x^2+1) - 1 = x^4 + 3x^2 + 1$$

Part (ii):

$$(f \circ g)(-1) = f(g(-1)) = f((-1)^2 + (-1) + 1) = f(-1) = 2$$

$f(h(a)) = 2 \Rightarrow f(1+a) = 2 \Rightarrow$

$(1+a)^2 + 1 = 2 \Rightarrow (a^2 + 2a + 1) + 1 = 2$

$\Rightarrow a^2 + 2a = 0 \Rightarrow a(a+2) = 0$

$\Rightarrow a = 0, -2$ (two answers for a)

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solution to question 4

$$\lim_{x \to 1} \sqrt[3]{\sin^2\left(\frac{x-1}{x^2-1}\right)} = \sqrt[3]{\lim \sin^2\left(\frac{x-1}{x^2-1}\right)}$$

Continuity of the root function

$$= \sqrt[3]{\sin^2\left(\lim_{x \to 1} \frac{x-1}{x^2-1}\right)}$$

→ Continuity of the sin function

$$= \sqrt[3]{\sin^2\left(\lim \frac{x-1}{(x-1)(x+1)}\right)}$$

$$= \sqrt[3]{\sin^2\left(\lim \frac{1}{x+1}\right)}$$

$$= \sqrt[3]{\sin^2\left(\frac{1}{2}\right)}$$