Lab 4 Solutions

1. By calculating the left-hand derivative and the right-hand derivative of the function

$$f(1) = \begin{cases} -\frac{1}{3} + x^2 & x < 1 \\ \frac{2}{3} & x = 1 \\ \frac{2}{3}x^3 & x > 1 \end{cases}$$

at $x = 1$ show that $f'(1)$ exists.

Solution:

$$\begin{aligned} f'_{-}(1) = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0^-} \frac{\left\{-\frac{1}{3}+(1+h)^2\right\}-\left\{\frac{2}{3}\right\}}{h} \\ &= \lim_{h\to 0^-} \frac{\left\{-\frac{1}{3}+(1+2h+h^2)\right\}-\left\{\frac{2}{3}\right\}}{h} \\ &= \lim_{h\to 0^-} \frac{\left\{\frac{2}{3}+2h+h^2\right\}-\left\{\frac{2}{3}\right\}}{h} \\ &= \lim_{h\to 0^-} \frac{2h+h^2}{h} \\ &= \lim_{h\to 0^-}(2+h) = 2 \end{aligned}$$

And

$$f'_+(1) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{\left\{\frac{2}{3}(1+h)^3\right\}-\left\{\frac{2}{3}\right\}}{h}$$

$$= \lim_{h\to 0^+} \frac{\left\{\frac{2}{3}(1+3h+3h^2+h^3)\right\}-\left\{\frac{2}{3}\right\}}{h}$$

$$= \lim_{h\to 0^+} \frac{\left\{\frac{2}{3}+2h+2h^2+\frac{2}{3}h^3\right\}-\left\{\frac{2}{3}\right\}}{h}$$

$$= \lim_{h\to 0^+} \frac{2h+2h^2+\frac{2}{3}h^3}{h}$$

$$= \lim_{h\to 0^+}(2+2h+\tfrac{2}{3}h^2) = 2$$

$$f'_-(1) = f'_+(1) \quad \Rightarrow \quad f'(1) \ \text{ exists}$$

2. Use the <u>definition</u> of derivative to calculate the derivative of the function $f(x) = \sqrt{x^2-1}$ at the point $x = 2$

<u>**Solution**</u>:

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\sqrt{(2+h)^2-1}-\sqrt{3}}{h} = \lim_{h\to 0} \frac{\sqrt{(4+4h+h^2)-1}-\sqrt{3}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3+4h+h^2}-\sqrt{3}}{h} = \lim_{h\to 0} \left(\frac{\sqrt{3+4h+h^2}-\sqrt{3}}{h}\right)\left(\frac{\sqrt{3+4h+h^2}+\sqrt{3}}{\sqrt{3+4h+h^2}+\sqrt{3}}\right)$$

$$= \lim_{h\to 0} \frac{(\sqrt{3+4h+h^2}\,)^2-(\sqrt{3})^2}{h(\sqrt{3+4h+h^2}+\sqrt{3})} = \lim_{h\to 0} \frac{(3+4h+h^2)-(3)}{h(\sqrt{3+4h+h^2}+\sqrt{3})} = \lim_{h\to 0} \frac{4h+h^2}{h(\sqrt{3+4h+h^2}+\sqrt{3})}$$

$$= \lim_{h\to 0} \frac{4+h}{(\sqrt{3+4h+h^2}+\sqrt{3})} = \frac{4+0}{\sqrt{3+0+0}+\sqrt{3}} == \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}}$$

3. Find the equation of the tangent line at the point $(0\,,1)$ on the graph of the function $y = x + \sqrt{x^2+1}$

Solution:

$$y = x + (x^2+1)^{\frac{1}{2}} \quad \Rightarrow \quad y' = 1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)$$

$$\Rightarrow \quad y' = 1 + \frac{x}{\sqrt{x^2+1}} \overset{\text{substitute } x=0}{\Longrightarrow} \quad \text{(slope)} \;\; m = 1 + \frac{0}{\sqrt{0+1}} = 1$$

$$\text{tangent line in general} \;\; y - y_0 = m(x - x_0) \quad \Rightarrow \quad y - 1 = 1(x-0)$$

$$\Rightarrow \quad \boxed{y = x + 1}$$

4. Solve questions 22 , 23 , and 25 of section 3.7 of the textbook. Hint: Before any attempt to differentiation do not forget to write the root functions in the form of power functions.

Solution for question 22:

$$y = \left(\frac{2-x}{2+x}\right)^{\frac{1}{4}}$$

$$\begin{aligned}
\Rightarrow \quad y' &= \tfrac{1}{4}\left(\tfrac{2-x}{2+x}\right)^{-\frac{3}{4}}\left(\tfrac{2-x}{2+x}\right)' \\
&= \tfrac{1}{4}\left(\tfrac{2-x}{2+x}\right)^{-\frac{3}{4}} \tfrac{\{2-x\}'\{2+x\}-\{2-x\}\{2+x\}'}{(2+x)^2} \\
&= \tfrac{1}{4}\left(\tfrac{2-x}{2+x}\right)^{-\frac{3}{4}} \tfrac{\{-1\}\{2+x\}-\{2-x\}\{1\}}{(2+x)^2} \\
&= \tfrac{1}{4}\left(\tfrac{2-x}{2+x}\right)^{-\frac{3}{4}} \tfrac{-4}{(2+x)^2} \\
&= \tfrac{-1}{(2-x)^{\frac{3}{4}}(2+x)^{\frac{5}{4}}} \qquad \checkmark
\end{aligned}$$

Solution for question 23:

$$
\begin{aligned}
y' &= \Big\{(x^3-2x^2)^3\Big\}'\Big\{(x^4-2x)^5\Big\} + \Big\{(x^3-2x^2)^3\Big\}\Big\{(x^4-2x)^5\Big\}' \\
&= \Big\{3(x^3-2x^2)^2\ (x^3-2x^2)'\Big\}\Big\{(x^4-2x)^5\Big\} + \Big\{(x^3-2x^2)^3\Big\}\Big\{5(x^4-2x)^4\ (x^4-2x)'\Big\} \\
&= \Big\{3(x^3-2x^2)^2\ (3x^2-4x)\Big\}\Big\{(x^4-2x)^5\Big\} + \Big\{(x^3-2x^2)^3\Big\}\Big\{5(x^4-2x)^4\ (4x^3-2)\Big\} \\
&\overset{\text{factorization}}{=} (x^3-2x^2)^2\,(x^4-2x)^4\Big\{3(3x^2-4x)\ (x^4-2x) + 5(x^3-2x^2)\ (4x^3-2)\Big\} \\
&\overset{\text{expansion}}{=} (x^3-2x^2)^2\,(x^4-2x)^4\,(29x^6-52x^5-28x^3+44x^2) \\
&\overset{\text{factorization}}{=} (x^3-2x^2)^2\,(x^4-2x)^4\,x^2\,(29x^4-52x^3-28x+44) \qquad \checkmark
\end{aligned}
$$

Solution for question 25:

$$
y = \frac{x(1-x^2)^{\frac{1}{2}}}{(3+x)^{\frac{1}{3}}}
$$

$$\Rightarrow \quad y' = \frac{\left\{x(1-x^2)^{\frac{1}{2}}\right\}'\left\{(3+x)^{\frac{1}{3}}\right\}-\left\{x(1-x^2)^{\frac{1}{2}}\right\}\left\{(3+x)^{\frac{1}{3}}\right\}'}{\left\{(3+x)^{\frac{1}{3}}\right\}^2}$$

$$= \frac{\left\{\{x\}'(1-x^2)^{\frac{1}{2}}+x\left\{(1-x^2)^{\frac{1}{2}}\right\}'\right\}\left\{(3+x)^{\frac{1}{3}}\right\}-\left\{x(1-x^2)^{\frac{1}{2}}\right\}\left\{(3+x)^{\frac{1}{3}}\right\}'}{(3+x)^{\frac{2}{3}}}$$

$$= \frac{\left\{\{1\}(1-x^2)^{\frac{1}{2}}+x\left\{\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\right\}\right\}\left\{(3+x)^{\frac{1}{3}}\right\}-\left\{x(1-x^2)^{\frac{1}{2}}\right\}\left\{\frac{1}{3}(3+x)^{-\frac{2}{3}}\right\}}{(3+x)^{\frac{2}{3}}}$$

$$= \frac{\left\{(1-x^2)^{\frac{1}{2}}-x^2(1-x^2)^{-\frac{1}{2}}\right\}\left\{(3+x)^{\frac{1}{3}}\right\}-\left\{x(1-x^2)^{\frac{1}{2}}\right\}\left\{\frac{1}{3}(3+x)^{-\frac{2}{3}}\right\}}{(3+x)^{\frac{2}{3}}}$$

$$= \frac{\left\{(1-x^2)^{-\frac{1}{2}}\left\{(1-x^2)-x^2\right\}\right\}\left\{(3+x)^{\frac{1}{3}}\right\}-\left\{x(1-x^2)^{\frac{1}{2}}\right\}\left\{\frac{1}{3}(3+x)^{-\frac{2}{3}}\right\}}{(3+x)^{\frac{2}{3}}}$$

$$= \frac{\left\{(1-x^2)^{-\frac{1}{2}}(1-2x^2)\right\}\left\{(3+x)^{\frac{1}{3}}\right\}-\left\{x(1-x^2)^{\frac{1}{2}}\right\}\left\{\frac{1}{3}(3+x)^{-\frac{2}{3}}\right\}}{(3+x)^{\frac{2}{3}}}$$

$$= \frac{\left(\frac{1-2x^2}{\sqrt{1-x^2}}\right)\left(\sqrt[3]{3+x}\right)-\left(x\sqrt{1-x^2}\right)\left(\frac{1}{3\sqrt[3]{(3+x)^2}}\right)}{\sqrt[3]{(3+x)^2}}$$

Now by multiplying the numerator and the denominator by $3\sqrt{1-x^2}\sqrt[3]{(3+x)^2}$ we get:

$$= \frac{3(1-2x^2)(3+x)-x(1-x^2)}{3\sqrt{1-x^2}\sqrt[3]{(3+x)^4}}$$

$$= \frac{-5x^3-18x^2+2x+9}{3\sqrt{1-x^2}\sqrt[3]{(3+x)^4}} \qquad \checkmark$$