Let us examine the behavior of the function

$$f(x) = \frac{x^2 - 1}{x - 1} \qquad x \neq 1$$

for the points close to $x = 1$. As you see in the following table as we approach the point $x = 1$ the values of the function approach the value $L = 2$. We describe this issue using the notation $\lim\limits_{x \to 1} \frac{x^2-1}{x-1} = 2$

point x	value of the function (x^2-1)/(x-1)
0.988	1.988
0.989	1.989
0.99	1.99
0.991	1.991
0.992	1.992
0.993	1.993
0.994	1.994
0.995	1.995
0.996	1.996
0.997	1.997
0.998	1.998
0.999	1.999

point x	value of the function (x^2-1)/(x-1)
1.012	2.012
1.011	2.011
1.01	2.01
1.009	2.009
1.008	2.008
1.007	2.007
1.006	2.006
1.005	2.005
1.004	2.004
1.003	2.003
1.002	2.002
1.001	2.001

Example: Evaluate $\lim_{x\to 1}(3x^2 - x + 1)$

Solution: As x gets close and closer to the value 1 , the function values get arbitrarily close to 3. Therefore we can say that $\lim_{x\to 1} 3x^2 - x + 1 = 3$.

Discussion: When we approach the point $a = 0$ the values of the function $f(x) = \frac{1}{x^2}$ become very large , and in fact the values of the function become arbitrarily large when we approach the point $a = 0$; see the graph. We describe this situation by writing

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

Note that the ∞ here is just a symbol and is not a number. So in this case we say that the limit does not exist, however we know that the values of the function get arbitrarily large when we approach $a = 0$. The values of function $f(x) = \frac{-1}{|x+1|}$ get arbitrarily large negative values when we approach the point $a = -1$; we describe this situation by writing

$$\lim_{x\to -1} \frac{-1}{|x+1|} = -\infty$$

By looking at the graphs of these two functions, one can see that the line $x = 0$ is the vertical asymptote for both graphs.

Example (section 2.2 exercise 26): Find the limit $\lim_{x\to a} \frac{|x-a|}{x^2-2ax+a^2}$, where a is any number.

Solution:

$$= \lim_{x\to a} \frac{|x-a|}{(x-a)^2} = \lim_{x\to a} \frac{|x-a|}{|x-a|^2} = \lim_{x\to a} \frac{1}{|x-a|} = \infty$$

Theorem (algebraic properties of the limit):

Suppose that the limits $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$ exist. Then

(i) $\lim_{x\to a}\{f(x) + g(x)\} = A + B$, that is

$$\lim_{x\to a}\{f(x) + g(x)\} = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

(ii) $\lim_{x\to a}\{f(x)-g(x)\} = A - B$, that is

$$\lim_{x\to a}\{f(x)-g(x)\} = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

(iii) $\lim_{x\to a} f(x)\, g(x) = A\, B$, that is

$$\lim_{x\to a} f(x)\, g(x) = \lim_{x\to a} f(x) \;\; \lim_{x\to a} g(x)$$

(iv) $\lim_{x\to a} cf(x) = cA$, c being an arbitrary constant , i.e.

$$\lim_{x\to a} c\, f(x) = c \lim_{x\to a} f(x)$$

(v) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{A}{B}$ provided that $B \neq 0$, that is

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \qquad\qquad \text{if } B \neq 0$$

Example (technique of rationalization): Evaluate $\lim_{x\to 0} \frac{\sqrt{2x+1}-\sqrt{3x+1}}{x}$

Solution: We cannot use the above theorem for this example because the divisor's limit is zero. Since the dividend has a limit of zero too , we use the "rationalization" technique:

$$\begin{aligned}
&= \lim_{x\to 0} \left\{ \frac{\sqrt{2x+1}-\sqrt{3x+1}}{x} \; \frac{\sqrt{2x+1}+\sqrt{3x+1}}{\sqrt{2x+1}+\sqrt{3x+1}} \right\} \\
&= \lim_{x\to 0} \left\{ \frac{(2x+1)-(3x+1)}{x(\sqrt{2x+1}+\sqrt{3x+1})} \right\} \qquad \text{using the rule} \;\; (a-b)(a+b) = a^2 - b^2 \\
&= \lim_{x\to 0} \left\{ \frac{(-x)}{x(\sqrt{2x+1}+\sqrt{3x+1})} \right\} = \lim_{x\to 0} \left\{ \frac{(-1)}{(\sqrt{2x+1}+\sqrt{3x+1})} \right\} = \frac{-1}{2}
\end{aligned}$$

Example (section 2.1 exercise 49): Assuming $a \neq 0$, use the rationalization technique to evaluate the limit

$$\lim_{x\to 0} \frac{\sqrt{x^2+a^2}-\sqrt{2x^2+a^2}}{\sqrt{3x^2+4}-\sqrt{2x^2+4}}$$

Solution:

$$= \lim_{x\to 0}\left(\frac{\sqrt{x^2+a^2}-\sqrt{2x^2+a^2}}{\sqrt{3x^2+4}-\sqrt{2x^2+4}}\right)\left(\frac{\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}}{\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}}\right)\left(\frac{\sqrt{3x^2+4}+\sqrt{2x^2+4}}{\sqrt{3x^2+4}+\sqrt{2x^2+4}}\right)$$

$$= \lim_{x\to 0}\left(\frac{(x^2+a^2)-(2x^2+a^2)}{(3x^2+4)-(2x^2+4)}\right)\left(\frac{1}{\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}}\right)\left(\frac{\sqrt{3x^2+4}+\sqrt{2x^2+4}}{1}\right)$$

$$= \lim_{x\to 0}\left(\frac{-x^2}{x^2}\right)\left(\frac{1}{\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}}\right)\left(\frac{\sqrt{3x^2+4}+\sqrt{2x^2+4}}{1}\right)$$

$$= \lim_{x\to 0}(-1)\left(\frac{1}{\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}}\right)\left(\frac{\sqrt{3x^2+4}+\sqrt{2x^2+4}}{1}\right)$$

$$= (-1)\left(\frac{1}{2\sqrt{a^2}}\right)\left(\frac{2\sqrt{4}}{1}\right) = \frac{-2}{|a|}$$

Example: The functions $\sqrt{1-x}-3$ and $2+\sqrt[3]{x}$ become zero at the point $x=-8$. Factor them out in the form $(x+8)g(x)$ and then find the limit $\lim_{x\to -8}\frac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}$.

Solution: **Step 1**.

$$\sqrt{1-x}-3 = \frac{(\sqrt{1-x}-3)(\sqrt{1-x}+3)}{\sqrt{1-x}+3} = \frac{(1-x)-9}{\sqrt{1-x}+3} = \frac{-x-8}{\sqrt{1-x}+3} = \frac{-(x+8)}{\sqrt{1-x}+3}$$

$$\text{here } g(x) = \frac{-1}{\sqrt{1-x}+3}$$

Step 2. Using the identity $a^3+b^3=(a+b)(a^2-ab+b^2)$ we have:

$$\begin{aligned}
2+\sqrt[3]{x} &= \frac{(2+\sqrt[3]{x})\left\{(2)^2-(2)(\sqrt[3]{x})+(\sqrt[3]{x})^2\right\}}{\left\{(2)^2-(2)(\sqrt[3]{x})+(\sqrt[3]{x})^2\right\}} \\
&= \frac{(2+\sqrt[3]{x})\left\{4-(2)(\sqrt[3]{x})+(\sqrt[3]{x})^2\right\}}{\left\{4-(2)(\sqrt[3]{x})+(\sqrt[3]{x})^2\right\}} \\
&= \frac{8-4\sqrt[3]{x}+2(\sqrt[3]{x})^2+4\sqrt[3]{x}-2(\sqrt[3]{x})^2+x}{\left\{4-(2)(\sqrt[3]{x})+(\sqrt[3]{x})^2\right\}} \\
&= \frac{8+x}{\left\{4-(2)(\sqrt[3]{x})+(\sqrt[3]{x})^2\right\}}
\end{aligned}$$

Step 3.

$$\lim_{x\to-8}\frac{\sqrt{1-x}-3}{2+\sqrt[3]{x}} = \lim_{x\to-8}\frac{\frac{-(x+8)}{\sqrt{1-x}+3}}{\frac{8+x}{\left\{4-(2)(\sqrt[3]{x})+(\sqrt[3]{x})^2\right\}}} = -\lim_{x\to-8}\frac{\left\{4-(2)(\sqrt[3]{x})+(\sqrt[3]{x})^2\right\}}{\sqrt{1-x}+3} = -\frac{12}{6} = -2$$

Example: Find the limit $\lim\limits_{x\to2^-}\frac{x^3-x-6}{|x-2|}$

Solution:

$$= \lim_{x\to2^-}\frac{x^3-x-6}{-(x-2)} = \lim_{x\to2^-}\frac{(x-2)(x^2+2x+3)}{-(x-2)} = \lim_{x\to2^-}(x^2+2x+3) = 11$$

Sometimes we are only allowed to approach a point from one side: for instance for the function $\sqrt{x}$ we can approach the point $a=0$ from the right only because the function is undefined on the left-hand side of that point. The right-hand limit at a point a is denoted by $\lim\limits_{x\to a^+}$ and the left-hand limit is denoted by $\lim\limits_{x\to a^-}$. We have a similar list of algebraic properties of the right-hand limit and the left-hand limit.

Example (exercise 18 section 2.2): Evaluate $\lim\limits_{x\to 0} \frac{\sqrt{1+x}-1}{x^2}$

Solution:

$$= \lim_{x\to 0} \left\{ \frac{\sqrt{1+x}-1}{x^2} \; \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} \right\} = \lim_{x\to 0} \frac{(1+x)-1}{x^2(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{x}{x^2(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{1}{x(\sqrt{1+x}+1)}$$

But we have

$$\lim_{x\to 0^-} \frac{1}{x(\sqrt{1+x}+1)} = -\infty \qquad \text{and} \qquad \lim_{x\to 0^+} \frac{1}{x(\sqrt{1+x}+1)} = \infty$$

so the limit does not exist.

Theorem (algebraic properties of the right-hand limit):

Suppose that the limits $\lim\limits_{x\to a^+} f(x) = A$ and $\lim\limits_{x\to a^+} g(x) = B$ exist. Then

(i) $\lim\limits_{x\to a^+} \{f(x) + g(x)\} = A + B$, that is

$$\lim_{x\to a^+} \{f(x) + g(x)\} = \lim_{x\to a^+} f(x) + \lim_{x\to a^+} g(x)$$

(ii) $\lim\limits_{x\to a^+} \{f(x) - g(x)\} = A - B$, that is

$$\lim_{x\to a^+} \{f(x) - g(x)\} = \lim_{x\to a^+} f(x) - \lim_{x\to a^+} g(x)$$

(iii) $\lim\limits_{x\to a^+} f(x)\, g(x) = A\, B$, that is

$$\lim_{x\to a^+} f(x)\, g(x) = \lim_{x\to a^+} f(x) \; \lim_{x\to a^+} g(x)$$

(iv) $\lim\limits_{x\to a^+} cf(x) = cA$, c being an arbitrary constant , i.e.

$$\lim_{x\to a^+} c\, f(x) = c \lim_{x\to a^+} f(x)$$

(v) $\lim\limits_{x\to a^+} \frac{f(x)}{g(x)} = \frac{A}{B}$, that is

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a^+} f(x)}{\lim\limits_{x\to a^+} g(x)}$$

Note: A similar list holds for the left-hand limit.

Theorem: A limit $\lim_{x\to a} f(x)$ exists if and only if both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal. In this case, the three limits are equal.

Discussion: Some functions have this interesting property at a point a that $\lim_{x\to a} f(x) = f(a)$. Such functions are said to be continuous at $x = a$. Geometrically, there is no hole in the graph of the function at the point $x = a$. Examples of continuous function are: polynomials , trigonometric function , exponential function, the log functions, and the power functions.

Example: Let k be any real number. Find the value of k such that the function

$$f(x) = \begin{cases} 2 + kx & x \geq 1 \\ 2k + 3x^2 & x < 1 \end{cases}$$

is continuous at $x = 1$.

Solution: We have $f(1) = 2 + k$ .For the continuity to hold we must have both

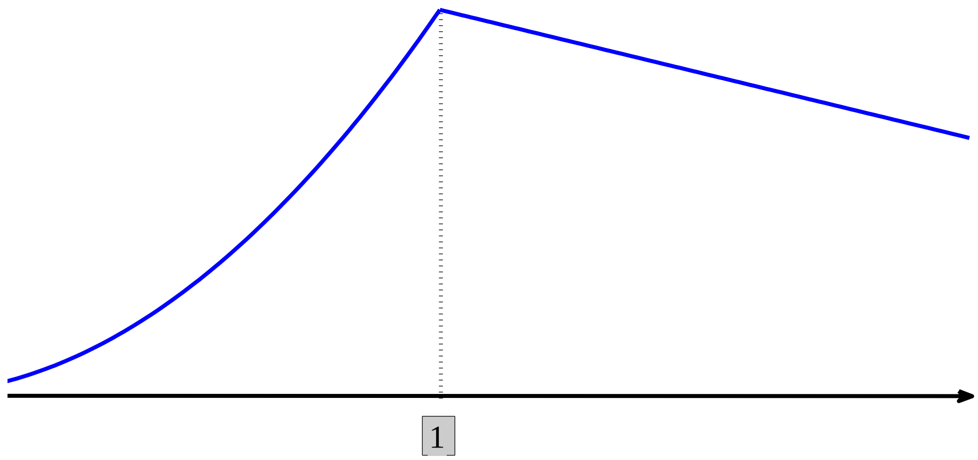
$$\begin{cases} \lim_{x\to 1^+} f(x) = f(1) \\ \lim_{x\to 1^-} f(x) = f(1) \end{cases}$$

The requirement $\lim_{x\to 1^+} f(x) = f(1)$ is equivalent to $\lim_{x\to 1^+} 2+kx = 2+k$ which is the same as the equality $2+k = 2+k$ which gives nothing. But the requirement $\lim_{x\to 1^-} f(x) = f(1)$ is equivalent to $\lim_{x\to 1^-} 2k + 3x^2 = 2 + k$ which is the same as $2k + 3 = 2 + k$ resulting in $\boxed{k = -1}$.

Note: For $k = -1$ this function becomes

$$f(x) = \begin{cases} 2 - x & x \geq 1 \\ -2 + 3x^2 & x < 1 \end{cases}$$

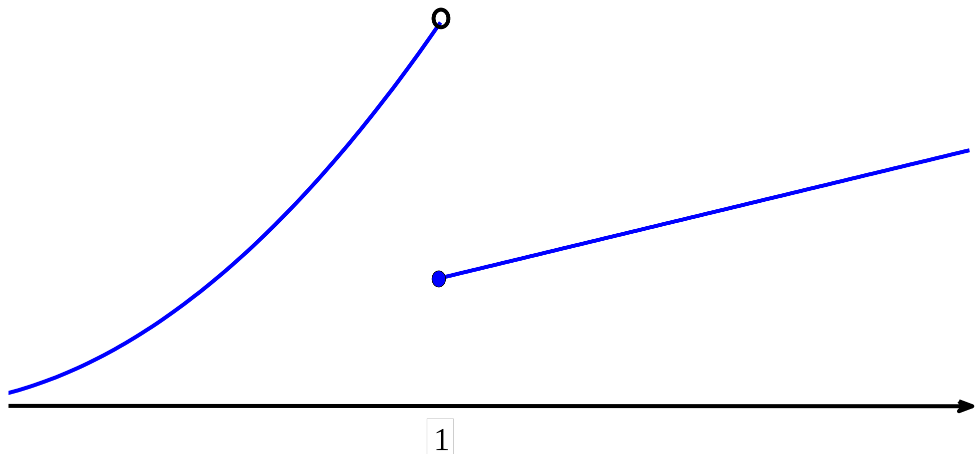
whose graph is



For $k = 1$ the function becomes

$$f(x) = \begin{cases} 2 + x & x \geq 1 \\ 2 + 3x^2 & x < 1 \end{cases}$$

whose graph is



Example: Find the values of a and b such that the function

$$f(x) = \begin{cases} a(x^2+1) - 2bx & x < 3 \\ 4 & x = 3 \\ 2a + 3bx & x > 3 \end{cases}$$

is continuous everywhere.

Solution: The function is continuous on the interval $(-\infty\,,\,3)$ because on this interval it is the same as the polynomial $a(x^2+1) - 2bx$ and we know that the polynomials are continuous. The function is continuous on the interval $(3\,,\,\infty)$ because it a polynomial there, namely $2a + 3bx$. The only point at which continuity is not guaranteed is the point $x = 3$ at which the rule of the function changes and the function behaves differently on both sides of this point. Now

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \left\{ a(x^2+1) - 2bx \right\} = 10a - 6b$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \left\{ 2a + 3bx \right\} = 2a + 9b$$

For this function to be continuous at $x = 3$ we must have

$$\begin{cases} \lim\limits_{x\to 3^-} f(x) = f(3) \\ \lim\limits_{x\to 3^+} f(x) = f(3) \end{cases} \Rightarrow \begin{cases} 10a - 6b = 4 \\ 2a + 9b = 4 \end{cases} \overset{simplifying}{\Rightarrow} \begin{cases} 5a - 3b = 2 \\ 2a + 9b = 4 \end{cases}$$

To solve this system, multiply the first row by 3 and then add it to the second row to get $17a = 10$ resulting in $\boxed{a = \frac{10}{17}}$. By putting this value of a into one of the equations one gets $\boxed{b = \frac{16}{51}}$.
