Limits at infinity

Section 2.3

Note: An example of limit at infinity that we frequently use is this:

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0$$

♣ **Theorem (algebraic properties of the limit at infinity)**: Suppose that the limits $\lim_{x\to\infty} f(x) = A$ and $\lim_{x\to\infty} g(x) = B$ exist. Then

(i) $\lim_{x\to\infty} \{f(x) + g(x)\} = A + B$, that is

$$\lim_{x\to\infty} \{f(x) + g(x)\} = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$$

(ii) $\lim_{x\to\infty} \{f(x) - g(x)\} = A - B$, that is

$$\lim_{x\to\infty} \{f(x) - g(x)\} = \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x)$$

(iii) $\lim_{x\to\infty} f(x)\, g(x) = A\, B$, that is

$$\lim_{x\to\infty} f(x)\, g(x) = \lim_{x\to\infty} f(x) \;\; \lim_{x\to\infty} g(x)$$

(iv) $\lim_{x\to\infty} cf(x) = cA$, c being an arbitrary constant , i.e.

$$\lim_{x\to\infty} c\, f(x) = c \lim_{x\to\infty} f(x)$$

(v) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{A}{B}$ provided that $B \neq 0$, that is

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} \qquad \text{if } \lim_{x\to\infty} g(x) \neq 0$$

Note: A similar theorem holds for the limits of the type $\lim\limits_{x\to-\infty}$

$$\boxed{\frac{0}{0} \text{ and } \frac{\infty}{\infty} \text{ and } \infty-\infty \text{ are indeterminate forms}}$$

Note: In calculating a "limit at infinity" if the limit is of the indeterminate form $\frac{\infty}{\infty}$ then the terms in the numerator and dominator with highest exponents are the dominant terms; see the following examples:

Example: Evaluate the limit $\lim_{x\to-\infty}\sqrt{(x+1)(x+2)}+x$

Solution: This limit is of the indeterminate form $\infty-\infty$. This is how it is calculated:

$$=\lim_{x\to-\infty}\frac{\left\{\sqrt{(x+1)(x+2)}+x\right\}\left\{\sqrt{(x+1)(x+2)}-x\right\}}{\left\{\sqrt{(x+1)(x+2)}-x\right\}}=\lim_{x\to-\infty}\frac{(x+1)(x+2)-x^2}{\sqrt{(x+1)(x+2)}-x}$$

$$=\lim_{x\to-\infty}\frac{3x+2}{\sqrt{(x+1)(x+2)}-x}=\lim_{x\to-\infty}\frac{3x+2}{\sqrt{x^2+3x+2}-x}=\lim_{x\to-\infty}\frac{3x+2}{\sqrt{x^2(1+\frac{3}{x}+\frac{2}{x^2})}-x}$$

$$=\lim_{x\to-\infty}\frac{3x+2}{|x|\sqrt{(1+\frac{3}{x}+\frac{2}{x^2})}-x}=\lim_{x\to-\infty}\frac{3x+2}{(-x)\sqrt{(1+\frac{3}{x}+\frac{2}{x^2})}-x}\qquad\text{as } x \text{ is eventually negative}$$

$$=\lim_{x\to-\infty}\frac{3+\frac{2}{x}}{-\sqrt{(1+\frac{3}{x}+\frac{2}{x^2})}-1}\qquad\text{by dividing by } x$$

$$=\frac{3}{-2}\qquad\checkmark$$

Example: Evaluate the limit $\lim_{x\to\infty}\sqrt{(x+1)(x+2)}+x$

Solution: The limit is equal to ∞ because both terms $\sqrt{(x+1)(x+2)}$ and x approach ∞. ✓

Example: Find the value of $\lim\limits_{x\to\infty}\frac{\sqrt{x^3-x^2+1}+\sqrt[3]{x^4+2}}{\sqrt[4]{x^6-3x^3+1}-\sqrt[5]{x^7+2x^4+1}}$

Solution:

$$=\lim_{x\to\infty}\frac{x^{\frac{3}{2}}\sqrt{1-\frac{1}{x}+\frac{1}{x^3}}+x^{\frac{4}{3}}\sqrt[3]{1+\frac{2}{x^4}}}{x^{\frac{3}{2}}\sqrt[4]{1-\frac{3}{x^3}+\frac{1}{x^6}}-x^{\frac{7}{5}}\sqrt[5]{1+\frac{2}{x^3}+\frac{1}{x^7}}}$$

$$= \lim_{x\to\infty} \frac{x^{\frac{3}{2}}\left\{\sqrt{1-\frac{1}{x}+\frac{1}{x^3}}+\frac{1}{x^{\frac{3}{2}-\frac{4}{3}}}\sqrt[3]{1+\frac{2}{x^4}}\right\}}{x^{\frac{3}{2}}\left\{\sqrt[4]{1-\frac{3}{x^3}+\frac{1}{x^6}}-\frac{1}{x^{\frac{3}{2}-\frac{7}{5}}}\sqrt[5]{1+\frac{2}{x^3}+\frac{1}{x^7}}\right\}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1-\frac{1}{x}+\frac{1}{x^3}}+\frac{1}{x^{\frac{1}{6}}}\sqrt[3]{1+\frac{2}{x^4}}}{\sqrt[4]{1-\frac{3}{x^3}+\frac{1}{x^6}}-\frac{1}{x^{\frac{1}{10}}}\sqrt[5]{1+\frac{2}{x^3}+\frac{1}{x^7}}} = \frac{1+0}{1-0} = 1 \qquad \checkmark$$

Definition: In either of the cases $\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$ we say that the horizontal line $y = L$ is a **horizontal asymptote** of the function f.

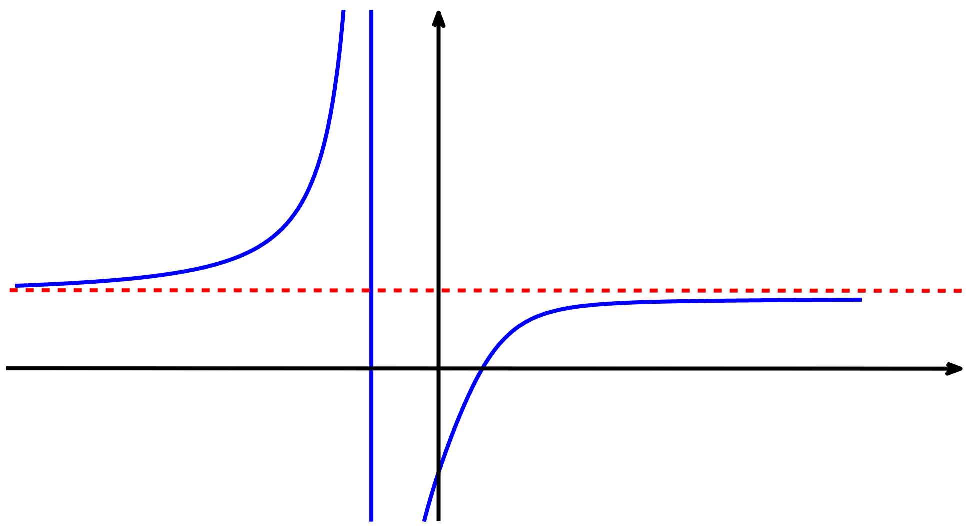
Note: For a rational function $R(x) = \frac{P(x)}{Q(x)}$ (P and Q being polynomials) , if the degrees of the numerator and denominator are equal, then there exists a horizontal asymptote; the following example describes this.

Example: Find the horizontal asymptotes of the function $y = \frac{2x^3-2x-1}{3x^3+x^2+x+1}$

solution:

$$\lim_{x\to\pm\infty} \frac{2x^3-2x-1}{3x^3+x^2+x+1} = \lim_{x\to\pm\infty} \frac{2-\frac{2}{x^2}-\frac{1}{x^3}}{3+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}} = \frac{2}{3}$$

So, the line $y = \frac{2}{3}$ is the only horizontal asymptote.



Note: The vertical asymptotes of a function $R(x) = \frac{P(x)}{Q(x)}$ (not necessarily a rational function) are amongst the lines $x = a$ where a is a root of the denominator. To decide which one of these is a vertical asymptote you should evaluate the limits $\lim_{x\to a^+} R(x)$ and $\lim_{x\to a^-} R(x)$ and if one of them equals either of $\pm\infty$ then we have vertical asymptote at that point a. Here are two examples:

Example (from the textbook): Find the vertical and horizontal asymptotes of the function $y = \frac{x^2-16}{x^2+x-6}$

solution: **Horizontal asymptotes**:

$$\lim_{x\to\infty} \frac{x^2-16}{x^2+x-6} = \lim_{x\to\infty} \frac{x^2\left(1-\frac{16}{x^2}\right)}{x^2\left(1+\frac{1}{x}-\frac{6}{x^2}\right)} = \lim_{x\to\infty} \frac{1-\frac{16}{x^2}}{1+\frac{1}{x}-\frac{6}{x^2}} = 1$$

The limit $\lim\limits_{x\to-\infty}$ would result in the same answer , so the line $y = 1$ is the only horizontal asymptote.

Vertical asymptotes:

$$x^2 + x - 6 = 0 \qquad \Rightarrow \qquad x = -3\,,\, 2$$

$$\lim_{x\to(-3)^+}\frac{x^2-16}{x^2+x-6}=\lim_{x\to(-3)^+}\frac{x^2-16}{(x+3)(x-2)}=\frac{(-5)}{(0^+)(-5)}=\infty$$

so the line $x=-3$ is a vertical asymptote.

$$\lim_{x\to2^+}\frac{x^2-16}{x^2+x-6}=\lim_{x\to2^+}\frac{x^2-16}{(x+3)(x-2)}=\frac{(-5)}{(5)(0^+)}=-\infty$$

so the line $x=2$ is a vertical asymptote.

The graph of this function is shown on page 124.

Example: Find the vertical asymptotes of the function $y=\frac{x^2-2x-3}{x^2+3x+2}$

Solution: The roots of the denominator are $x=-2\,,\,-1$

$$\lim_{x\to(-2)^+}\frac{x^2-2x-3}{x^2+3x+2}=\lim_{x\to(-2)^+}\frac{x^2-2x-3}{(x+2)(x+1)}=\frac{5}{(0^+)(3)}=\infty$$

So , the line $x=-2$ is a vertical asymptote.

$$\lim_{x\to(-1)^+}\frac{x^2-2x-3}{x^2+3x+2}=\lim_{x\to(-1)^+}\frac{(x+1)(x-3)}{(x+2)(x+1)}==\lim_{x\to(-1)^+}\frac{(x-3)}{(x+2)}=\frac{-4}{1}=-4$$

And the limit $\lim_{x\to(-1)^-}\frac{x^2-2x-3}{x^2+3x+2}$ would result in the same value of -4. Therefore , although the point $x=-1$ is a root of the denominator however we do not have a vertical asymptote there. So remember that

♣ a root of denominator is not necessarily a vertical asymptote ♣

Definition: If we have either $\lim\limits_{x\to-\infty}\Big\{f(x)-(ax+b)\Big\}=0$ or $\lim\limits_{x\to\infty}\Big\{f(x)-(ax+b)\Big\}=0$ we say that the line $y=ax+b$ is an **oblique asymptote** of the function f.

Example: Using the long division divide x^3 by x^2+x+1 :

$$x^3=(x^2+x+1)(x-1)+1$$

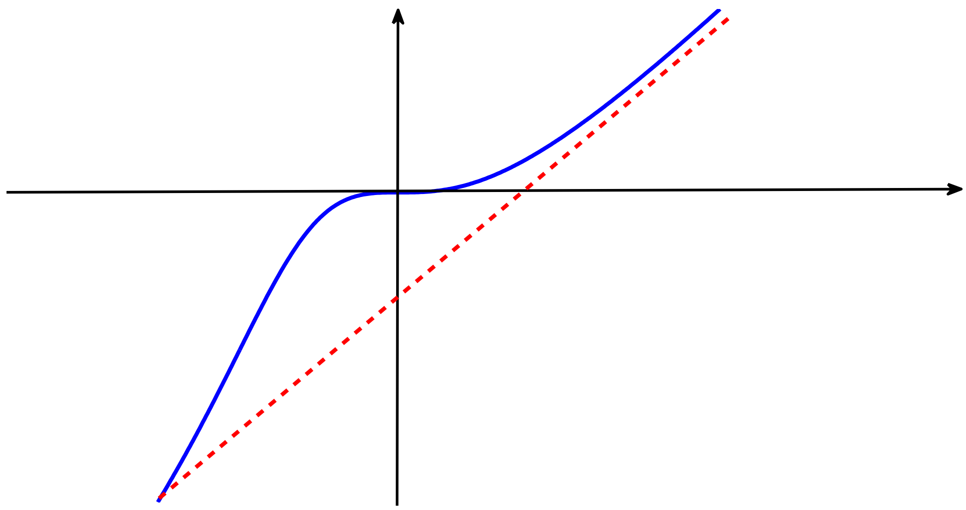
Divide both sides by x^2+x+1 :

$$\frac{x^3}{x^2+x+1}=x-1+\frac{1}{x^2+x+1}\qquad\Rightarrow\qquad\frac{x^3}{x^2+x+1}-(x-1)=\frac{1}{x^2+x+1}$$

Then

$$\lim_{x\to\pm\infty} \frac{x^3}{x^2+x+1} - (x-1) = \lim_{x\to\infty} \frac{1}{x^2+x+1} = 0$$

Therefore by definition, the line $y = x - 1$ is an oblique asymptote for the function $\frac{x^3}{x^2+x+1}$.



Note: In a rational function (i.e. the quotient of two polynomials) with the degree of numerator being one plus the degree of the denominator , we have an oblique asymptote. The example above shows how to find its oblique asymptote.

♣ **Sandwich Theorem**: Let lim denote any of the limits $\lim\limits_{x\to a}$, $\lim\limits_{x\to a^+}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to\infty}$, and $\lim\limits_{x\to-\infty}$. Let for the points close to the point where the limit is being calculated at we have $f(x) \le g(x) \le h(x)$ (so for example if the limit $\lim\limits_{x\to\infty}$ is being calculated then it is assumed that we have the inequalities $f(x) \le g(x) \le h(x)$ for all large x's). If under these assumptions we have $\lim f(x) = \lim h(x) = L$ then we have $\lim g(x) = L$

♣ **Theorem**: Let lim denote any of the limits $\lim\limits_{x\to a}$, $\lim\limits_{x\to a^+}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to\infty}$, and $\lim\limits_{x\to-\infty}$. If $\lim |f(x)| = 0$, then $\lim f(x) = 0$.

♣ **Theorem (zero-times-bounded Theorem)**: Let lim denote any of the limits $\lim_{x\to a}$, $\lim_{x\to a^+}$, $\lim_{x\to a^-}$, $\lim_{x\to\infty}$, and $\lim_{x\to-\infty}$. Let for the points close to the point where the limit is being calculated at the function f remains bounded. Suppose further that $\lim g(x) = 0$. Then $\lim f(x)g(x) = 0$.

Example: Evaluate

$$\lim_{x\to\infty} \frac{x^3 + x^2\sin(x^2+1)}{x^3+1}$$

Solution:

$$= \lim_{x\to\infty} \frac{x^3\left(1+\frac{1}{x}\sin(x^2+1)\right)}{x^3\left(1+\frac{1}{x^3}\right)} = \lim_{x\to\infty} \frac{1+\frac{1}{x}\sin(x^2+1)}{1+\frac{1}{x^3}} = \frac{1+0}{1+0} = 1$$

noting that the term $\frac{1}{x}$ tends to zero while $\sin(x^2+1)$ is bounded , therefore by an application of the zero-times-bounded theorem the limit of $\frac{1}{x}\sin(x^2+1)$ is zero.