Logarithmic Differentiation
Section 3.12

Important Note. When you see an expression involving exponents , multiplication, and division only, then use logarithmic differentiation . The following examples describe this method:

Example. Find $\frac{dy}{dx}$ if $y = (\sin x)^{\tan x}$

Solution.

$$\ln y = (\tan x)\ \ln(\sin x)$$

$$\overset{\text{differentiate}}{\Rightarrow} \quad \frac{y'}{y} = \Big\{\tan x\Big\}'\ \ln(\sin x) + (\tan x)\Big\{\ln(\sin x)\Big\}'$$

$$= \Big\{\sec^2 x\Big\}\ \ln(\sin x) + (\tan x)\Big\{\frac{(\sin x)'}{\sin x}\Big\}$$

$$= \Big\{\sec^2 x\Big\}\ \ln(\sin x) + (\tan x)\Big\{\frac{\cos x}{\sin x}\Big\}$$

$$= (\sec^2 x)\ \ln(\sin x) + 1$$

$$\Rightarrow \quad y' = y\Big\{(\sec^2 x)\ \ln(\sin x) + 1\Big\}$$

$$= (\sin x)^{\tan x}\Big\{(\sec^2 x)\ \ln(\sin x) + 1\Big\} \qquad \checkmark$$

Important Note. It is **wrong** to answer this question by using the power rule :

$$y' = (\tan x)(\sin x)^{\tan x - 1} \quad \text{this is wrong}$$

because here both the base and exponent are variable.

Important Note. When both the base and exponent are variables, then you must use logarithmic differentiation . When one of the base or the exponent is constant , then use the techniques we have learned before, such as the derivatives of the exponential functions or the (extended) power rule. Here is an example:

Example. Find $\frac{dy}{dx}$ if

(a) $y = 5^{\sin x}$ **use the differentiation rule for the exponential functions**

(b) $y = \tan^{8.3} x$ **use the power rule**

(c) $y = (\cos x)^x$ **use the logarithmic differentiation**

Solution of part (a).

$$y' = (\sin x)'\ 5^{\sin x} \ln(5) = (\cos x)\ 5^{\sin x} \ln(5) \qquad \checkmark$$

Solution of part (b).

$$y' = (8.3)(\tan x)^{7.3}(\tan x)' = 8.3(\tan x)^{7.3}(\sec^2 x) \qquad \checkmark$$

Solution of part (c).

$$\ln(y) = x\ln(\cos x)$$

$$\Rightarrow \quad \frac{y'}{y} = \{x\}'\ln(\cos x) + \{x\}\frac{(\cos x)'}{(\cos x)}$$
$$= \{1\}\ln(\cos x) + \{x\}\ \frac{-\sin x}{\cos x} = \ln(\cos x) - x\tan x$$

$$\Rightarrow \quad y' = \Big\{\ln(\cos x) - x\tan x\Big\}(\cos x)^x \qquad \checkmark$$

Example (section 3.12 exercise 5). Use logarithmic differentiation to find $f'(x)$ for

$$f(x) = \left(1 + \frac{1}{x}\right)^x$$

Solution.

$$f(x) = \left(1 + \frac{1}{x}\right)^x \quad \Rightarrow \quad \ln y = x \, \ln\left(1 + \frac{1}{x}\right)$$

$$\begin{aligned}
\tfrac{y'}{y} &= x' \ln\left(1 + \tfrac{1}{x}\right) + x \left\{\ln\left(1 + \tfrac{1}{x}\right)\right\}' \\
&= \ln\left(1 + \tfrac{1}{x}\right) + x \; \tfrac{\left(1+\frac{1}{x}\right)'}{\left(1+\frac{1}{x}\right)} \\
&= \ln\left(1 + \tfrac{1}{x}\right) + x \; \tfrac{\left(\frac{x+1}{x}\right)'}{\left(1+\frac{1}{x}\right)} \\
&= \ln\left(1 + \tfrac{1}{x}\right) + x \; \tfrac{\frac{(x+1)'(x)-(x+1)(x)'}{x^2}}{\left(\frac{x+1}{x}\right)} \\
&= \ln\left(1 + \tfrac{1}{x}\right) + x \; \tfrac{\frac{(x)-(x+1)}{x^2}}{\left(\frac{x+1}{x}\right)} \\
&= \ln\left(1 + \tfrac{1}{x}\right) + x \; \tfrac{\frac{-1}{x^2}}{\left(\frac{x+1}{x}\right)} \\
&= \ln\left(1 + \tfrac{1}{x}\right) - \tfrac{1}{x+1}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad y' &= y \left\{\ln\left(1 + \tfrac{1}{x}\right) - \tfrac{1}{x+1}\right\} \\
&= \left(1 + \tfrac{1}{x}\right)^x \left\{\ln\left(1 + \tfrac{1}{x}\right) - \tfrac{1}{x+1}\right\} \qquad \checkmark
\end{aligned}$$

Example (section 3.12 exercise 11). Use logarithmic differentiation to find $f'(x)$ for

$$f(x) = (x^2 + 3x^4)^3 (x^2 + 5)^4$$

Solution. Of course , one can use the product rule and the power rule to find the derivative, but the question has asked us to use the logarithmic differentiation technique.

$$\begin{aligned} & f(x) = (x^2 + 3x^4)^3 (x^2 + 5)^4 \\ \Rightarrow \quad & \ln y = 3\ln(x^2 + 3x^4) + 4\ln(x^2 + 5) \\ \tfrac{y'}{y} \quad &= 3\tfrac{(x^2+3x^4)'}{(x^2+3x^4)} + 4\tfrac{(x^2+5)'}{(x^2+5)} \\ &= 3\tfrac{2x+12x^3}{(x^2+3x^4)} + 4\tfrac{2x}{(x^2+5)} \\ &= \tfrac{6x+36x^3}{(x^2+3x^4)} + \tfrac{8x}{(x^2+5)} \\ &= \tfrac{(6x+36x^3)(x^2+5)+(8x)(x^2+3x^4)}{(x^2+3x^4)(x^2+5)} \\ &= \tfrac{2x(30x^4+97x^2+15)}{(x^2+3x^4)(x^2+5)} \end{aligned}$$

$$\Rightarrow \quad y' = y\,\frac{2x(30x^4+97x^2+15)}{(x^2+3x^4)(x^2+5)}$$

$$= (x^2+3x^4)^3(x^2+5)^4\,\frac{2x(30x^4+97x^2+15)}{(x^2+3x^4)(x^2+5)}$$

$$= 2x(x^2+3x^4)^2(x^2+5)^3(30x^4+97x^2+15) \qquad \checkmark$$

Use logarithmic differentiation when the function is made up of multiplications, products, and powers (including root functions) , or when both the base and exponent are variable

Example. Differentiate the function $y = \sqrt[3]{\frac{(x^2-1)}{x^{\frac{3}{7}}(e^x+1)}}$.

Solution.

$$y = \left(\frac{(x^2-1)}{x^{\frac{3}{7}}(e^x+1)}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \ln(y) = \frac{1}{3}\ln\left(\frac{(x^2-1)}{x^{\frac{3}{7}}(e^x+1)}\right)$$

$$= \frac{1}{3}\Big\{\ln(x^2-1) - \frac{3}{7}\ln x - \ln(e^x+1)\Big\}$$

$$\Rightarrow \qquad \frac{y'}{y} = \frac{1}{3}\left\{\frac{(x^2-1)'}{(x^2-1)} - \frac{3}{7}\frac{1}{x} - \frac{(e^x+1)'}{(e^x+1)}\right\}$$

$$= \frac{1}{3}\left\{\frac{2x}{x^2-1} - \frac{3}{7x} - \frac{e^x}{e^x+1}\right\}$$

$$\Rightarrow \quad y' = \frac{y}{3}\left\{\frac{2x}{x^2-1} - \frac{3}{7x} - \frac{e^x}{e^x+1}\right\}$$

$$= \frac{1}{3}\sqrt[3]{\frac{(x^2-1)}{x^{\frac{3}{7}}(e^x+1)}}\left\{\frac{2x}{x^2-1} - \frac{3}{7x} - \frac{e^x}{e^x+1}\right\}$$