Midterm Test 1

DATE: October 15, 2015
TIME: 75 minutes
COURSE: MATH 1510
EXAMINERS: H. Farhadi and S. Tsaturian

NAME (Write with capital letters): ____________________

STUDENT No. ____________________

CHECK YOUR SECTION :

- ❑ A01 8:30–9:20 AM MWF (208 ARMS) H. Farhadi
- ❑ A02 1:00–2:15 PM TR (206 HUMAN ECOLOGY) S. Tsaturian

You may use back of pages should you need more space.

Please do not write in the boxes below. Go directly to question 1.

Q1 [4]	**Q2** [5]	**Q3** [9]	**Q4** [6]	**Q5** [5]	**Q6** [6]	**Q7** [5]	Total (out of 40)

[4] 1. Find the equation of the tangent line at the point $(3\,,\,3)$ on the graph of the function $y = 5 - \sqrt{x+1}$

Solution.

$$\frac{dy}{dx} = -\frac{1}{2}(x+1)^{-\frac{1}{2}} = -\frac{1}{2\sqrt{x+1}}$$

By substituting $x = 3$ we get the slope:

$$\text{slope} = -\frac{1}{2\sqrt{3+1}} = -\frac{1}{4}$$

Then the equation of the tangent line:

$$(y-3) = -\frac{1}{4}(x-3) \quad \Rightarrow \quad x + 4y = 15$$

[5] 2. Apply the differentiation rules to find the derivative of this function:

$$f(x) = \left(\frac{x^3-1}{2x^3+1}\right)^4 + \sqrt{x+1}$$

Do not simplify your answer.

Solution.

$$f'(x) = \left\{\left(\frac{x^3-1}{2x^3+1}\right)^4\right\}' + \left\{(x+1)^{\frac{1}{2}}\right\}' = 4\left(\frac{x^3-1}{2x^3+1}\right)^3 \left(\frac{x^3-1}{2x^3+1}\right)'$$

$$= 4\left(\frac{x^3-1}{2x^3+1}\right)^3 \frac{(x^3-1)'(2x^3+1)-(x^3-1)(2x^3+1)'}{(2x^3+1)^2} + \frac{1}{2}(x+1)^{-\frac{1}{2}}$$

$$= 4\left(\frac{x^3-1}{2x^3+1}\right)^3 \frac{(3x^2)(2x^3+1)-(x^3-1)(6x^2)}{(2x^3+1)^2} + \frac{1}{2}(x+1)^{-\frac{1}{2}}$$

3. Calculate the following limits (part (b) is on the next page):

[6] (a) $$\lim_{x\to 3} \frac{\sqrt{3x+7}-4}{\sqrt{x+1}-2}$$

Solution.

$$= \lim_{x\to 3} \frac{(\sqrt{3x+7}-4)(\sqrt{3x+7}+4)(\sqrt{x+1}+2)}{(\sqrt{x+1}-2)(\sqrt{3x+7}+4)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{\left((\sqrt{3x+7})^2-4^2\right)(\sqrt{x+1}+2)}{\left((\sqrt{x+1})^2-2^2\right)(\sqrt{3x+7}+4)}$$

$$= \lim_{x\to 3} \frac{((3x+7)-16)(\sqrt{x+1}+2)}{((x+1)-4)(\sqrt{3x+7}+4)}$$

$$= \lim_{x\to 3} \frac{3(x-3)(\sqrt{x+1}+2)}{(x-3)(\sqrt{3x+7}+4)}$$

$$= \lim_{x\to 3} \frac{3(\sqrt{x+1}+2)}{(\sqrt{3x+7}+4)}$$

$$= \frac{3(\sqrt{3+1}+2)}{(\sqrt{9+7}+4)} = \frac{3}{2}$$

[3] (b) $\displaystyle\lim_{x\to\infty} \frac{2x^3+\cos 5x}{x^3}$

Solution. **First method**:

$$= \lim_{x\to\infty} \frac{x^3\left(2+\frac{1}{x^3}\cos 5x\right)}{x^3}$$

$$= \lim_{x\to\infty} \left(2+\frac{1}{x^3}\cos 5x\right)$$

Since the function $\frac{1}{x^3}$ tends to zero and the function $\cos 5x$ is bounded, by a theorem the product $\frac{1}{x^3}\cos 5x$ tends to zero , and therefore we will have the limit equal to :

$$= (2+0) = 2$$

Second method:

$$\frac{2x^3+\cos 5x}{x^3} = \frac{2x^3}{x^3} + \frac{\cos 5x}{x^3} = 2 + \frac{\cos 5x}{x^3}$$

Now note that

$$-1 \le \cos 5x \le 1 \quad \Rightarrow \quad -\frac{1}{x^3} \le \frac{\cos 5x}{x^3} \le \frac{1}{x^3}$$

Since

$$\begin{cases} \lim_{x\to\infty} \left(-\frac{1}{x^3}\right) = 0 \\ \\ \lim_{x\to\infty} \frac{1}{x^3} = 0 \end{cases}$$

then by the Squeeze Theorem (Sandwich Theorem) we have

$$\lim_{x\to\infty} \frac{\cos 5x}{x^3} = 0$$

and then

$$= \lim_{x\to\infty} \frac{x^3\left(2+\frac{1}{x^3}\cos 5x\right)}{x^3} = 2 + \lim_{x\to\infty} \frac{\cos 5x}{x^3} = 2+0 = 2$$

[6] 4. Find the horizontal asymptotes of the function

$$f(x) = \frac{\sqrt{3x^2+5}}{4x+1}$$

Solution.

$$\lim_{x\to\pm\infty} \frac{\sqrt{3x^2+5}}{4x+1} = \lim_{x\to\pm\infty} \frac{\sqrt{x^2\left(3+\frac{5}{x^2}\right)}}{4x+1}$$

$$= \lim_{x\to\pm\infty} \frac{|x|\sqrt{\left(3+\frac{5}{x^2}\right)}}{4x+1}$$

$$= \lim_{x\to\infty} \frac{(\pm x)\sqrt{\left(3+\frac{5}{x^2}\right)}}{4x+1}$$

$$= \lim_{x\to\infty} \frac{(\pm x)\sqrt{\left(3+\frac{5}{x^2}\right)}}{x\left(4+\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\pm\sqrt{\left(3+\frac{5}{x^2}\right)}}{\left(4+\frac{1}{x}\right)}$$

$$= \frac{\pm\sqrt{(3+0)}}{(4+0)} = \pm\frac{\sqrt{3}}{4}$$

So , the lines $y = \pm\frac{\sqrt{3}}{4}$ are the only horizontal asymptotes.

[5] 5. Use the <u>definition</u> of derivative to calculate the derivative of the function $f(x) = \sqrt{2x+3}$ at the point $x = 0$

<u>Solution</u>.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{2h+3} - 3}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{2h+3} - 3)(\sqrt{2h+3} + 3)}{h(\sqrt{2h+3} + 3)}$$

$$= \lim_{h\to 0} \frac{(2h+3) - 3}{h(\sqrt{2h+3} + 3)}$$

$$= \lim_{h\to 0} \frac{2h}{h(\sqrt{2h+3} + 3)}$$

$$= \lim_{h\to 0} \frac{2}{(\sqrt{2h+3} + 3)}$$

$$= \frac{2}{\sqrt{(2)(0)+3} + 3} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

[6] 6. By calculating the left-hand derivative and the right-hand derivative of the function

$$f(x) = \begin{cases} x^3 + 1 & x < 0 \\ 1 & x = 0 \\ x^3 - x^2 + 1 & x > 0 \end{cases}$$

at $x = 0$ show that $f'(0)$ exists.

Solution. **First method**:

$$f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{(h^3+1) - 1}{h}$$
$$= \lim_{h\to 0^-} \frac{h^3}{h} = \lim_{h\to 0^-} h^2 = 0$$

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{(h^3 - h^2 + 1) - 1}{h}$$
$$= \lim_{h\to 0^+} \frac{h^3 - h^2}{h} = \lim_{h\to 0^-} (h^2 - h) = 0$$

Therefore

$$f'_-(0) = f'_+(0) \quad \Rightarrow \quad f'(0) \text{ exists}$$

Second method: Te function can be expressed in the form:

$$f(x) = \begin{cases} x^3 + 1 & x \le 0 \\ x^3 - x^2 + 1 & x \ge 0 \end{cases}$$

The function coincides with the function $x^3 + 1$ on the left-hand side of 0 , therefore

$$f'_-(0) = \left.\frac{d(x^3+1)}{dx}\right|_{x=0} = (3x^2)\big|_{x=0} = 0$$

The function coincides with the function $x^3 - x^2 + 1$ on the right-hand side of 0 , therefore

$$f'_+(0) = \left.\frac{d(x^3 - x^2 + 1)}{dx}\right|_{x=0} = (3x^2 - 2x)\big|_{x=0} = 0$$

Therefore

$$f'_-(0) = f'_+(0) \quad \Rightarrow \quad f'(0) \text{ exists}$$

[5] 7. Consider the function

$$f(x) = \begin{cases} 1 - kx^2 & x \le 2 \\ k + x & x > 2 \end{cases}$$

Find the value of k for which the function f is continuous at $x = 2$. Justify your answer by computing appropriate limits.

Solution.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (1 - kx^2) = 1 - 4k$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (k + x) = k + 2$$

For the continuity we must have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) \quad \Rightarrow \quad 1 - 4k = k + 2 \quad \Rightarrow \quad k = -\frac{1}{5}$$

Scrap Paper