How to show that a function is one-to-one

Example. Show that the function $f(x) = \frac{x-3}{2x+1}$ is one-to-one on its domain $D = \{x \quad : \quad x \neq -\frac{1}{2}\}$.

Solution. **Step 1**. We must show that

$$f(s) = f(t) \qquad \Rightarrow \qquad s = t$$

Here is how:

$$f(s) = f(t) \qquad \Rightarrow \qquad \frac{s-3}{2s+1} = \frac{t-3}{2t+1} \qquad \Rightarrow \qquad (s-3)(2t+1) = (t-3)(2s+1) \qquad \Rightarrow$$

$$2st + s - 6t - 3 = 2st + t - 6s - 3 \qquad \Rightarrow \qquad s - 6t = t - 6s \qquad \Rightarrow \qquad s + 6s = t + 6t$$

$$\Rightarrow \qquad 7s = 7t \qquad \Rightarrow \qquad s = t \qquad \checkmark$$

How to find the inverse function

Example. Find the inverse function of the function $f(x) = \dfrac{x+1}{x-2}$.

Solution.

$$
\begin{aligned}
y = \frac{x+1}{x-2} \quad &\overset{\textbf{find x}}{\Longrightarrow} \quad (x-2)y = x+1 \\
&\Longrightarrow \quad xy - 2y - (x+1) = 0 \\
&\Longrightarrow \quad x(y-1) = 2y+1 \\
&\Longrightarrow \quad x = \tfrac{2y+1}{y-1} \\
&\Longrightarrow \quad f^{-1}(y) = \tfrac{2y+1}{y-1}
\end{aligned}
$$

Example (section 1.6 exercise 24). Find the inverse of the function

$$y = x^2 - x \qquad x \geq \frac{1}{2}$$

Solution.

$$
\begin{aligned}
y = x^2 - x \quad &\overset{\textbf{find x}}{\Longrightarrow} \quad x^2 - x - y = 0 \\
&\Longrightarrow \quad x = \tfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \tfrac{1 \pm \sqrt{1+4y}}{2}
\end{aligned}
$$

But to see which one is the answer, we write:

$$x - \frac{1}{2} = \pm \frac{\sqrt{1+4y}}{2}$$

But according to the condition $x \geq \frac{1}{2}$, we have $x - \frac{1}{2} \geq 0$, therefore we must take the positive sign. So

$$x = \frac{1+\sqrt{1+4y}}{2} \quad \Rightarrow \quad f^{-1}(y) = \frac{1+\sqrt{1+4y}}{2} \quad \checkmark$$

Solution (a second method). Complete the square:

$$y = \underbrace{x^2 - x + \frac{1}{4}}_{\text{complete square}} - \frac{1}{4} = (x - \frac{1}{2})^2 - \frac{1}{4} \quad \Rightarrow \quad y + \frac{1}{4} = (x - \frac{1}{2})^2 \quad \overset{\textbf{take square root}}{\Longrightarrow}$$

$$\sqrt{y + \frac{1}{4}} = |x - \frac{1}{2}| \quad \overset{x \geq \frac{1}{2}}{\Longrightarrow} \quad \sqrt{y + \frac{1}{4}} = x - \frac{1}{2} \quad \Rightarrow \quad x = \frac{1}{2} + \sqrt{y + \frac{1}{4}}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{1}{2} + \sqrt{y + \frac{1}{4}} \quad \checkmark$$

Example (section 1.6 exercise 26). Find the inverse function for $y = \dfrac{e^x}{1 + 2e^x}$

Solution.

$$y = \frac{e^x}{1 + 2e^x} \quad \Rightarrow \quad y(1 + 2e^x) = e^x \quad \Rightarrow \quad y(1 + 2e^x) = e^x \quad \Rightarrow$$

$$e^x(2y - 1) + y = 0 \quad \Rightarrow \quad e^x(2y - 1) = -y \quad \Rightarrow \quad e^x = \frac{-y}{2y - 1} \quad \Rightarrow$$

$$x = \ln\left(\frac{-y}{2y - 1}\right) \quad \Rightarrow \quad f^{-1}(y) = \ln\left(\frac{-y}{2y - 1}\right)$$

How to solve equations and inequalities involving logarithmic functions

Example (section 1.6 exercise 52-a). Solve the equation $e^{2x+3} - 7 = 0$ for x.

Solution.

$$e^{2x+3} - 7 = 0 \quad \Rightarrow \quad e^{2x+3} = 7 \quad \Rightarrow \quad 2x + 3 = \ln(7) \quad \Rightarrow \quad 2x = \ln(7) - 3$$

$$\Rightarrow \quad x = \frac{\ln(7) - 3}{2} \quad \checkmark$$

Example (section 1.6 exercise 53-b). Solve the equation $\ln(x) + \ln(x-1) = 1$ for x.

Solution.

$$\ln(x) + \ln(x-1) = 1 \quad \Rightarrow \quad \ln(x(x-1)) = 1 \quad \Rightarrow \quad x(x-1) = e^1 = e$$

$$\Rightarrow \quad x^2 - x = e \quad \Rightarrow \quad x^2 - x - e = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1+4e}}{2} \quad \checkmark$$

Example (section 1.6 exercise 54-a). Solve the equation $\ln(\ln x) = 1$

Solution.

$$\ln(\ln x) = 1 \quad \overset{\text{apply exponential}}{\Longrightarrow} \quad \ln(x) = e^1 = e \quad \overset{\text{apply exponential}}{\Longrightarrow} \quad x = e^e$$

Example. Solve the equation $3^{2x-1} = 4^{x+2}$

Solution.

$$\begin{aligned} 3^{2x-1} = 4^{x+2} \quad \overset{\text{apply ln}}{\Longrightarrow} \quad (2x-1)\ln(3) &= (x+2)\ln(4) \\ 2x\ln(3) - \ln(3) &= x\ln(4) + 2\ln(4) \\ 2x\ln(3) - x\ln(4) &= 2\ln(4) + \ln(3) \\ x(2\ln 3 - \ln 4) &= 2\ln 4 + \ln 3 \\ x &= \frac{2\ln 4 + \ln 3}{2\ln 3 - \ln 4} \end{aligned}$$

Example. Solve the equation $\log_3(3x^2)^{\frac{1}{4}} = 3$

Solution.

$$\begin{aligned} \log_3(3x^2)^{\frac{1}{4}} = 3 \quad \Longrightarrow \quad \frac{1}{4}\log_3(3x^2) &= 3 \\ \log_3(3x^2) &= 12 \\ 3x^2 &= \log_3^{-1}(12) \\ 3x^2 &= 3^{12} \\ x^2 &= 3^{11} \\ |x| &= 3^{\frac{11}{2}} \\ x &= \pm 3^{\frac{11}{2}} \end{aligned}$$

Example. Solve the inequality $\log_{0.3}|4-3x| < 0$

Solution.

$$\log_{0.3}|4-3x| < 0 \quad \overset{\text{apply } \log_{0.3}^{-1}}{\Longrightarrow} \quad |4-3x| > \log_{0.3}^{-1}(0) \qquad \log_{0.3} \text{ is a decreasing function}$$

$$\Longrightarrow \quad |4-3x| > (0.3)^0 \quad \Longrightarrow \quad |4-3x| > 1 \quad \Longrightarrow \quad \pm(4-3x) > 1$$

$$\Longrightarrow \quad \begin{cases} 4-3x > 1 \\ \text{or} \\ -(4-3x) > 1 \end{cases} \Rightarrow \begin{cases} 4-3x > 1 \\ \text{or} \\ 4-3x < -1 \end{cases} \Rightarrow \begin{cases} 3 > 3x \\ \text{or} \\ 5 < 3x \end{cases}$$

$$\Rightarrow \quad \begin{cases} 1 > x \\ \text{or} \\ \frac{5}{3} < x \end{cases} \quad \Rightarrow \quad x \in (-\infty\ ,\ 1) \cup (\frac{5}{3}\ ,\ +\infty)$$

Example (section 1.6 exercise 56). Solve the inequality $1 < e^{3x-1} < 2$

Solution.

$$1 < e^{3x-1} < 2 \quad \xRightarrow{\text{apply } \ln} \quad \ln(1) < 3x - 1 < \ln(2) \quad \Rightarrow \quad 0 < 3x - 1 < \ln(2)$$

$$\Rightarrow \quad 1 < 3x < 1 + \ln(2) \quad \Rightarrow \quad \frac{1}{3} < x < \frac{1 + \ln(2)}{3}$$

Example (section 1.6 exercise 57).

(a) What is the domain of $f(x) = \ln(e^x - 3)$?

(b) Find f^{-1} and its domain.

Solution to part (a).

$$x \in D_f \quad \Rightarrow \quad e^x - 3 > 0 \quad \Rightarrow \quad e^x > 3 \quad \xRightarrow{\text{apply } \ln} \quad x > \ln(3) \quad \Rightarrow \quad D_f = (\ln(3)\ ,\ \infty)$$

Solution to part (b).

$$\ln(e^x - 3) = y \quad \xRightarrow{\text{apply exp}} \quad e^x - 3 = e^y \quad \Rightarrow \quad e^x = e^y + 3 \quad \xRightarrow{\text{apply ln}}$$

$$x = \ln(e^y + 3) \quad \Rightarrow \quad f^{-1}(y) = \ln(e^y + 3) \qquad \checkmark$$

$y \in D_{f^{-1}} \quad \Rightarrow \quad e^y + 3 > 0 \quad \Rightarrow \quad e^y > -3$ But this holds as $e^y > 0$. Therefore, all y's are in the domain of f^{-1}: $\quad D_{f^{-1}} = (-\infty\ ,\ \infty)$