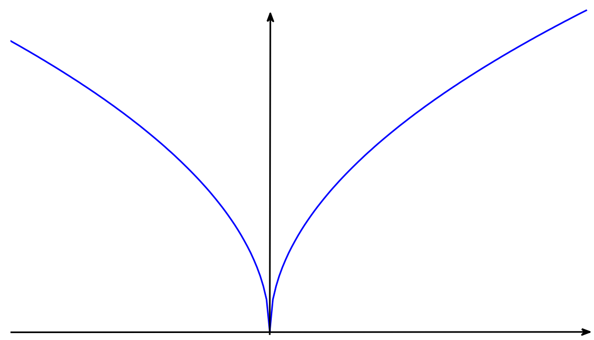
Symmetry (odd and even functions)

The graphs of the functions

$$y = x^2$$
$$y = \sqrt{|x|}$$

are symmetric about the y-axis:



Such functions are called **even**. Note that for such functions the domain is symmetric about the y-axis and that in the domain we have $f(-x) = f(x)$.

The graph of the function $y = x^5 - x$ is symmetric with respect to the origin. Such a function is called an **odd function**. Note that for such a function the domain is symmetric with respect to the origin and that in the domain we have $f(-x) = -f(x)$. So:

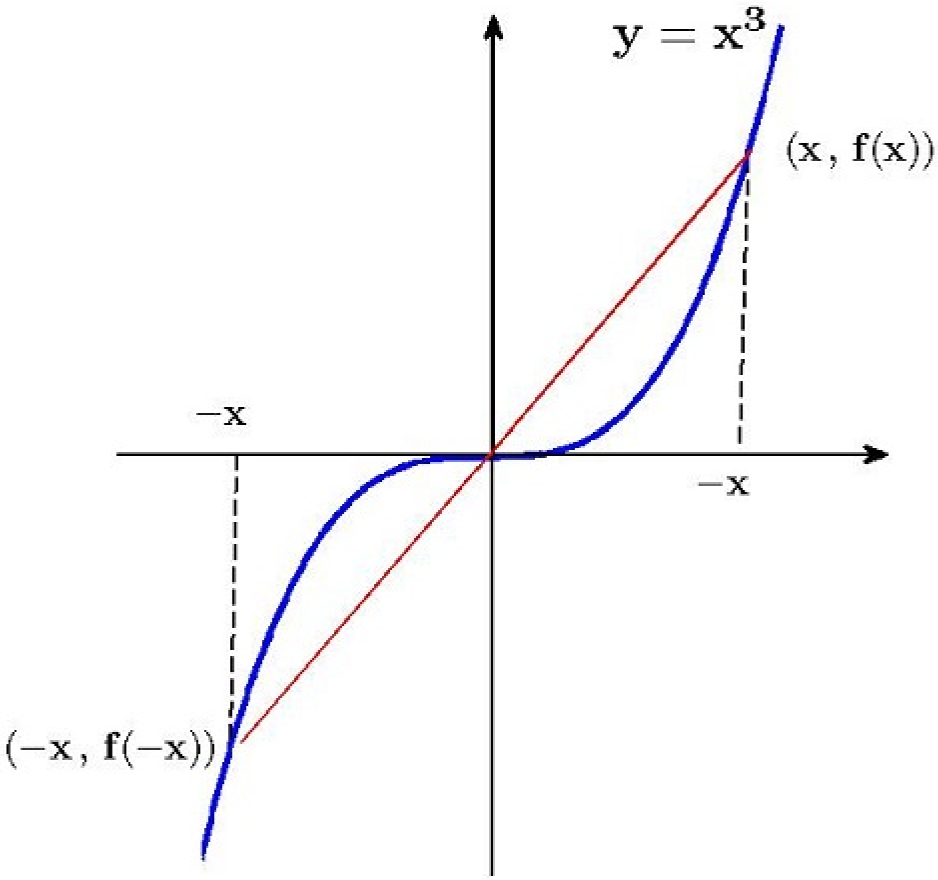
Definition. A function is called **even** if its domain is symmetric with respect to the origin and has this property:

$$f(-x) = f(x)$$

Definition. A function is called **odd** if its domain is symmetric with respect to the origin and satisfies:

$$f(-x) = -f(x)$$

- These are the ways we algebraically verify that a function is even or odd or neither.



Example. The function $f(x) = \sqrt{|x|}$ is even because:

$$f(-x) = \sqrt{|-x|} = \sqrt{|x|} = f(x)$$

Example. The function $f(x) = x^5 - x$ is an odd function because:

$$f(-x) = (-x)^5 - (-x) = -x^5 + x = -(x^5 - x) = -f(x)$$

Example. The function $f(x) = x^2 + x$ is neither odd nor even:

$$f(-x) = (-x)^2 + (-x) = x^2 - x$$

which is not equal to either of $f(x) = x^2 + x$ and $-f(x) = -x^2 - x$. Or you may check that its graph is neither symmetric with respect to the y-axis nor being symmetric with respect to the origin.

Note. Here we solved all parts of exercise 5 of part II of Lab 1 in class.

Fact. The trigonometric functions $\sin x$, $\tan x$, $\cot x$, and $\csc x = \frac{1}{\sin x}$ are odd functions. The functions $\cos x$ and $\sec x = \frac{1}{\cos x}$ (which is the reciprocal of $\sin x$) are even functions.

$$(\textbf{odd functions}) \begin{cases} \sin(-x) = -\sin x \\ \tan(-x) = -\tan x \\ \cot(-x) = -\cot(x) \\ \csc(-x) = -\csc(x) \end{cases} \qquad (\textbf{even functions}) \begin{cases} \cos(-x) = \cos x \\ \sec(-x) = \sec x \end{cases}$$

Example. Identify the function $f(x) = \sin(x+2) + |2x-1|$ as even, odd, or neither.

Solution:

$$\begin{aligned} f(-x) &= \sin(-x+2) + |2(-x)-1| \\ &= \sin(-x+2) + |-2x-1| \\ &= \sin(-x+2) + |2x+1| \end{aligned}$$

So

$$\begin{cases} f(-x) \neq f(x) \\ f(-x) \neq -f(x) \end{cases}$$

So this function is neither odd nor even.

Example. Identify the function $f(x) = 2\ \sin(5x) + \frac{3x^7}{1+x^6}$ as even, odd, or neither.

Solution: Change x to $-x$ to get $f(-x)$ and then simplify as much as you can:

$$\begin{aligned}
f(-x) &= 2\ \sin(-5x) + \tfrac{3(-x)^7}{1+(-x)^6} \\
&= 2\Big(-\sin(5x)\Big) + \tfrac{-3x^7}{1+x^6} \\
&= -\Big\{2\ \sin(5x) + \tfrac{3x^7}{1+x^6}\Big\} \\
&= -f(x)
\end{aligned}$$

So this function is an odd function.

Example. Identify the function $f(x) = 3\cos(2x) + \frac{3x^3}{2x^5+x}$ as even, odd, or neither.

Solution: Change x to $-x$ to get $f(-x)$ and then simplify as much as you can:

$$\begin{aligned} f(-x) &= 3\cos(-2x) + \tfrac{3(-x)^3}{2(-x)^5+(-x)} \\ &= 3\cos(2x) + \tfrac{-3x^3}{-2x^5-x} \\ &= \textbf{cancel out a negative sign from the numerator and denominator} \\ &= 3\cos(2x) + \tfrac{3x^3}{2x^5+x} \\ &= f(x) \end{aligned}$$

So this function is an even function.

Example. Identify the function $f(x) = 3\tan(5x) + \dfrac{2x}{-2x^2+3x+4}$ as even, odd, or neither.

Solution: Change x to $-x$ to get $f(-x)$ and then simplify as much as you can:

$$\begin{aligned}
f(-x) &= 3\tan(-5x) + \tfrac{2(-x)}{-2(-x)^2+3(-x)+4} \\
&= -3\tan(5x) + \tfrac{-2x}{-2x^2-3x+4} \\
&= -3\tan(5x) + \tfrac{-2x}{-2x^2-3x+4} \\
&= -3\tan(5x) - \tfrac{2x}{-2x^2-3x+4} \\
&\neq \begin{cases} \mathbf{f(x)} = \mathbf{3}\tan(\mathbf{5x}) + \tfrac{\mathbf{2x}}{\mathbf{-2x^2+3x+4}} \\ \\ \mathbf{-f(x)} = \mathbf{-3}\tan(\mathbf{5x}) - \tfrac{\mathbf{2x}}{\mathbf{-2x^2+3x+4}} \qquad \text{the difference is in the term } 3x \end{cases}
\end{aligned}$$

So this function is neither odd nor even.