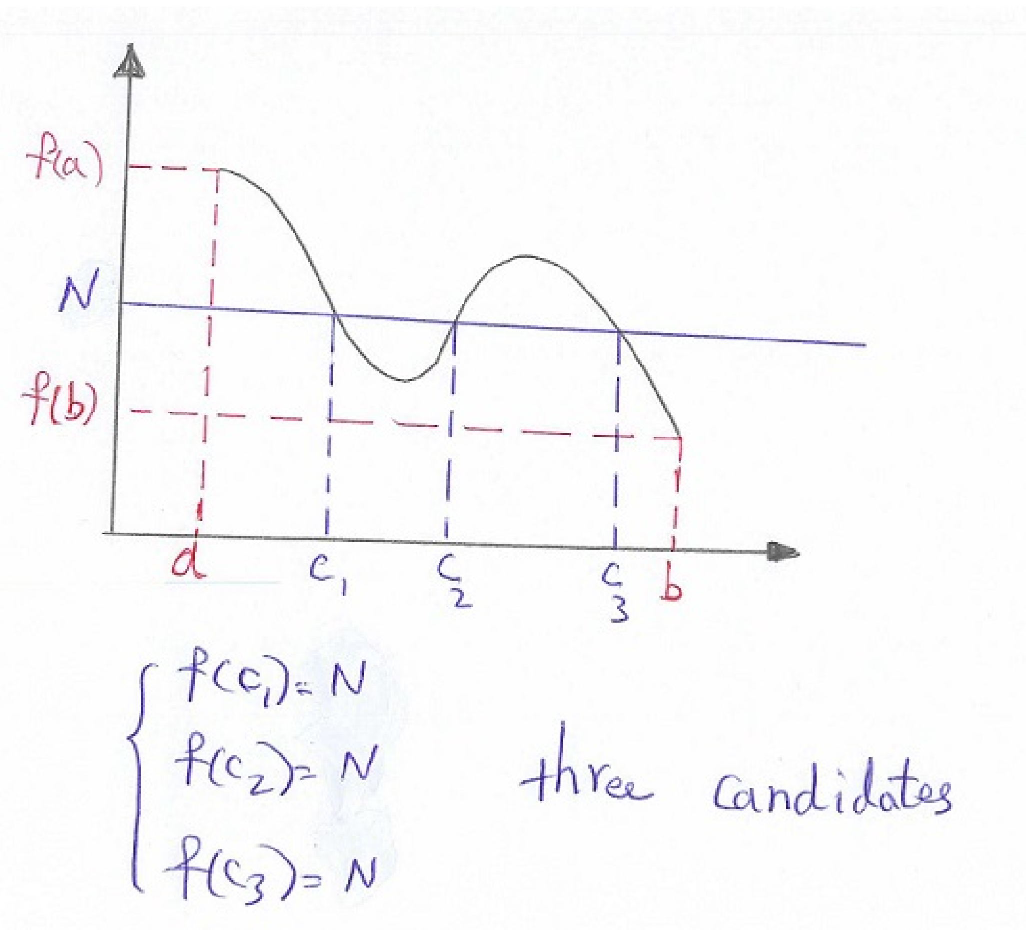
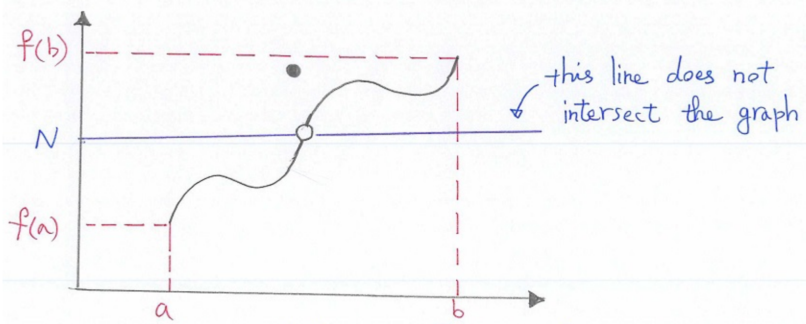
Intermediate Value Theorem

(from section 2.5)

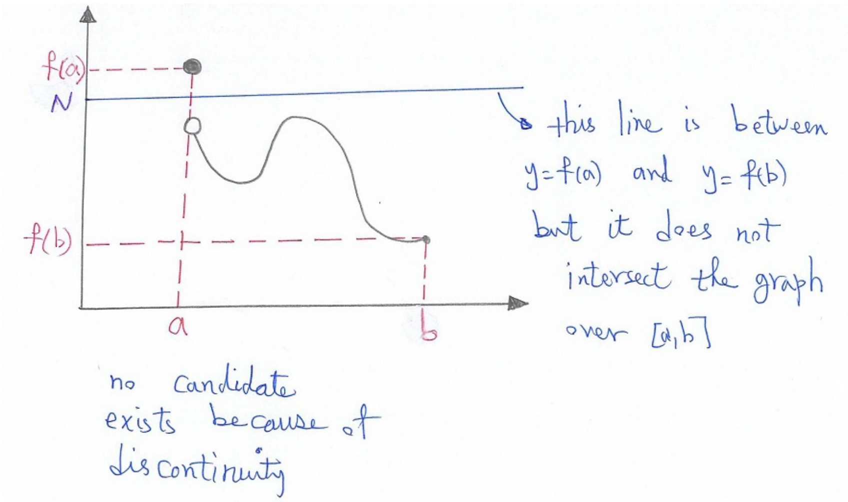
Theorem: Suppose that f is continuous on the interval $[a\,,\,b]$ (it is continuous on the path from a to b). If $f(a) \neq f(b)$ and if N is a number between $f(a)$ and $f(b)$ ($f(a) < N < f(b)$ or $f(b) < N < f(a)$), then there is number c in the open interval $a < c < b$ such that $f(c) = N$.

Note. This theorem says that any horizontal line between the two horizontal lines $y = f(a)$ and $y = f(b)$ intersects the graph of f somewhere between a and b. See figures 8 and 9 on page 126. Also see the following figures:





no candidate : discontinuity causes no candidate to exist.


f(a)
N
f(b)
a
b
this line is between
y=f(a) and y=f(b)
but it does not
intersect the graph
over [a,b]
no candidate
exists because of
discontinuity

Example (from the textbook). Use the Intermediate Value Theorem to show that there is root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

in the interval $[1\ ,\ 2]$.

Solution:

Consider the function $f(x) = 4x^3 - 6x^2 + 3x - 2$ over the closed interval $[1\ ,\ 2]$. **(this gets some marks)**.

The function f is a polynomial, therefore it is continuous over $[1\ ,\ 2]$ **(this gets some marks)**.

We have

$$\begin{cases} f(1) &= 4 - 6 + 3 - 2 = -1 \\ f(2) &= 32 - 24 + 6 - 2 = 12 \end{cases}$$

(this gets some marks)

(this gets some marks)

Since:

$$f(1) < 0 < f(2)$$ **(this gets some marks)**

by the Mean-Value-Theorem there exists a value c in the interval $(1\ ,\ 2)$ such that $f(c) = 0$, i.e. there is a solution for the equation $f(x) = 0$ in the interval $(1\ ,\ 2)$. **(this gets some marks)**.

Example (Midterm 1 - Fall 2016). Show that the equation $2 - e^x = 4x$ has a solution in the interval $[0, 1]$. Justify your answer.

Solution: We take everything to one side and set the equation in the equivalent form $4x - 2 + e^x = 0$. Consider the function $f(x) = 4x - 2 + e^x$. Equivalently, we must show that $f(x) = 0$ for some x in the interval $(0\,,\ 1)$ **(this gets some marks)**.

Because f is a difference of a polynomial and an exponential function, it is continuous everywhere, in particular, on the closed interval $[0\ ,\ 1]$ **(this gets some marks)**.

$$\begin{cases} f(0) &= 0 - 2 + e^0 = -2 + 1 = -1 & \textbf{(this gets some marks)} \\ f(1) &= 4 - 2 + e^1 = 2 + e \approx 2 + 2.718 = 4.718 & \textbf{(this gets some marks)} \end{cases}$$

Since:

$$f(0) < 0 < f(1) \qquad \textbf{(this gets some marks)}$$

by the Mean-Value-Theorem there exists a value c in the interval $(0\,,\,1)$ such that $f(c) = 0$, i.e. there is a solution for the equation $f(x) = 0$ in the interval $(0\,,\,1)$. **(this gets some marks)**

Example (exrcise 53 of section 2.5). Use the Intermediate Value Theorem to show that there is root of the equation

$$4x^4 + x - 3 = 0$$

in the interval $[1\ ,\ 2]$.

Solution:

Consider the function $f(x) = 4x^3 - 6x^2 + 3x - 2$ over the closed interval $[1\ ,\ 2]$. **(this gets some marks)**.

The function f is a polynomial, therefore it is continuous over $[1\ ,\ 2]$ **(this gets some marks)**.

We have

$$\begin{cases} f(1) &= 4 - 6 + 3 - 2 = -1 \qquad \textbf{(this gets some marks)} \\ f(2) &= 32 - 24 + 6 - 2 = 12 \qquad \textbf{(this gets some marks)} \end{cases}$$

Since:

$$f(1) < 0 < f(2) \qquad \textbf{(this gets some marks)}$$

by the Mean-Value-Theorem there exists a value c in the interval $(1\ ,\ 2)$ such that $f(c) = 0$, i.e. there is a solution for the equation $f(x) = 0$ in the interval $(1\ ,\ 2)$. **(this gets some marks)**.

Example (from the textbook exercises). Use the Intermediate Value Theorem to show that there is root of the equation

$$x^4 + x - 3 = 0$$

in the interval $[1\ ,\ 2]$.

Solution:

Consider the function $f(x) = x^4+x-3$ over the closed interval $[1\ ,\ 2]$. **(this gets some marks)**.

The function f is a polynomial, therefore it is continuous over $[1\ ,\ 2]$ **(this gets some marks)**.

We have

$$\begin{cases} f(1) &= 1 + 1 - 3 = -1 & \textbf{(this gets some marks)} \\ f(2) &= 16 + 2 - 3 = 15 & \textbf{(this gets some marks)} \end{cases}$$

Note that

$$f(1) < 0 < f(2) \qquad \textbf{(this gets some marks)}$$

So the number 0 is between two end values of f over the interval $[1\ ,\ 2]$, so by the Intermediate Value Theorem the value 0 must be covered by f over the interval $[1\ ,\ 2]$, i.e. there exists a value c in the interval $(1\ ,\ 2)$ such that $f(c) = 0$, i.e. there is a solution for the equation $x^4 + x - 3 = 0$ in the interval $(1\ ,\ 2)$ (that solution is actually the point c). **(this gets some marks)**.