Calculating Limits Using Rationalization Technique

Example (Midterm Exam - Winter 2014). Evaluate the $\lim_{x \to 25} \frac{5-\sqrt{x}}{25-x}$

Solution: This is of the ambiguous form $\frac{0}{0}$.

$$= \lim_{x \to 25} \frac{(5-\sqrt{x})}{(25-x)} \frac{(5+\sqrt{x})}{(5+\sqrt{x})}$$

$$= \lim_{x \to 25} \frac{(5)^2 - (\sqrt{x})^2}{(25-x)(5+\sqrt{x})} \qquad \text{using the rule } (a-b)(a+b) = a^2 - b^2$$

$$= \lim_{x \to 25} \frac{25-x}{(25-x)(5+\sqrt{x})}$$

$$= \lim_{x \to 25} \frac{\cancel{25-x}}{(\cancel{25-x})(5+\sqrt{x})}$$

$$= \lim_{x \to 25} \frac{1}{5+\sqrt{x}}$$

$$= \frac{1}{5+\sqrt{25}} = \frac{1}{5+5} = \frac{1}{10}$$

Example (Midterm Exam - Fall 2014). Evaluate the $\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x}$

Solution: This is of the ambiguous form $\frac{0}{0}$.

$$= \lim_{x\to 0} \frac{(\sqrt{x^2+1}-1)}{x} \frac{(\sqrt{x^2+1}+1)}{(\sqrt{x^2+1}+1)}$$

$$= \lim_{x\to 0} \frac{(\sqrt{x^2+1})^2-(1)^2}{x(\sqrt{x^2+1}+1)} \qquad \text{using the rule } (a-b)(a+b) = a^2 - b^2$$

$$= \lim_{x\to 0} \frac{(x^2+1)-1}{x(\sqrt{x^2+1}+1)}$$

$$= \lim_{x\to 0} \frac{x^2}{x(\sqrt{x^2+1}+1)}$$

$$= \lim_{x\to 0} \frac{x}{(\sqrt{x^2+1}+1)} \qquad \text{(after deleting } x\text{)}$$

$$= \frac{0}{\sqrt{0+1}+1} = \frac{0}{2} = 0$$

Example (From the textbook). Evaluate the $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

Solution: This is of the ambiguous form $\frac{0}{0}$.

$$= \lim_{t\to 0} \frac{(\sqrt{t^2+9}-3)}{t^2} \frac{(\sqrt{t^2+9}+3)}{(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{(\sqrt{t^2+9})^2-(3)^2}{t^2(\sqrt{t^2+9}+3)} \qquad \text{using the rule } (a-b)(a+b)=a^2-b^2$$

$$= \lim_{t\to 0} \frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{t^2}{t^2(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{1}{(\sqrt{t^2+9}+3)} \qquad \text{(after deleting } t^2\text{)}$$

$$= \frac{1}{\sqrt{0+9}+3} = \frac{1}{3+3} = \frac{1}{6}$$

Example (section 2.3 exercise 30). Evaluate the $\lim_{x\to -4} \frac{\sqrt{x^2+9}-5}{x+4}$

Solution: This is of the ambiguous form $\frac{0}{0}$.

$$= \lim_{x\to -4} \frac{(\sqrt{x^2+9}-5)}{x+4}\,\frac{(\sqrt{x^2+9}+5)}{(\sqrt{x^2+9}+5)}$$

$$= \lim_{x\to -4} \frac{(\sqrt{x^2+9})^2-(5)^2}{(x+4)(\sqrt{x^2+9}+5)} \qquad \text{using the rule } (a-b)(a+b)=a^2-b^2$$

$$= \lim_{x\to -4} \frac{(x^2+9)-25}{(x+4)(\sqrt{x^2+9}+5)}$$

$$= \lim_{x\to -4} \frac{x^2-16}{(x+4)(\sqrt{x^2+9}+5)}$$

$$= \lim_{x\to -4} \frac{(x-4)(x+4)}{(x+4)(\sqrt{x^2+9}+5)}$$

$$= \lim_{x\to -4} \frac{(x-4)\cancel{(x+4)}}{\cancel{(x+4)}(\sqrt{x^2+9}+5)}$$

$$= \lim_{x\to -4} \frac{(x-4)}{(\sqrt{x^2+9}+5)}$$

$$= \frac{-4-4}{\sqrt{16+9}+5} = \frac{-8}{5+5} = \frac{-4}{5}$$

Example . Evaluate $\lim_{x\to 0} \frac{\sqrt{2x+1}-\sqrt{3x+1}}{x}$

Solution: This is of the ambiguous form $\frac{0}{0}$.

$$= \lim_{x\to 0} \left\{ \frac{\sqrt{2x+1}-\sqrt{3x+1}}{x} \; \frac{\sqrt{2x+1}+\sqrt{3x+1}}{\sqrt{2x+1}+\sqrt{3x+1}} \right\}$$

$$= \lim_{x\to 0} \frac{(\sqrt{2x+1})^2-(\sqrt{3x+1})^2}{x(\sqrt{2x+1}+\sqrt{3x+1})} \qquad \text{using the rule } (a-b)(a+b)=a^2-b^2$$

$$= \lim_{x\to 0} \left\{ \frac{(2x+1)-(3x+1)}{x(\sqrt{2x+1}+\sqrt{3x+1})} \right\}$$

$$= \lim_{x\to 0} \left\{ \frac{-x}{x(\sqrt{2x+1}+\sqrt{3x+1})} \right\}$$

$$= \lim_{x\to 0} \left\{ \frac{-\cancel{x}}{\cancel{x}(\sqrt{2x+1}+\sqrt{3x+1})} \right\}$$

$$= \lim_{x\to 0} \left\{ \frac{-1}{(\sqrt{2x+1}+\sqrt{3x+1})} \right\} = \frac{-1}{2}$$

Example. Evaluate the limit

$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-\sqrt{2x^2+9}}{\sqrt{3x^2+4}-\sqrt{2x^2+4}}$$

Solution:

$$= \lim_{x\to 0} \left(\frac{\sqrt{x^2+9}-\sqrt{2x^2+9}}{\sqrt{3x^2+4}-\sqrt{2x^2+4}}\right)\left(\frac{\sqrt{x^2+9}+\sqrt{2x^2+9}}{\sqrt{x^2+9}+\sqrt{2x^2+9}}\right)\left(\frac{\sqrt{3x^2+4}+\sqrt{2x^2+4}}{\sqrt{3x^2+4}+\sqrt{2x^2+4}}\right)$$

$$= \lim_{x\to 0} \left(\frac{(x^2+9)-(2x^2+9)}{(3x^2+4)-(2x^2+4)}\right)\left(\frac{1}{\sqrt{x^2+9}+\sqrt{2x^2+9}}\right)\left(\frac{\sqrt{3x^2+4}+\sqrt{2x^2+4}}{1}\right)$$

$$= \lim_{x\to 0} \left(\frac{-x^2}{x^2}\right)\left(\frac{1}{\sqrt{x^2+9}+\sqrt{2x^2+9}}\right)\left(\frac{\sqrt{3x^2+4}+\sqrt{2x^2+4}}{1}\right)$$

$$= \lim_{x\to 0}(-1)\left(\frac{1}{\sqrt{x^2+9}+\sqrt{2x^2+9}}\right)\left(\frac{\sqrt{3x^2+4}+\sqrt{2x^2+4}}{1}\right)$$

$$= (-1)\left(\frac{1}{2\sqrt{9}}\right)\left(\frac{\sqrt{4}+\sqrt{4}}{1}\right) = \frac{-2}{3}$$