Limits involving Trigonometric Functions
(from section 3.3)

In the following examples we use the following two formulas (which you can use in exams freely):

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \mathbf{1}}$$

$$\boxed{\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = \mathbf{1}}$$

Important Note: When calculating the limits involving trigonometric functions, always look for an expression like $\frac{\sin x}{x}$ or $\frac{x}{\sin x}$ if $x \to 0$ because in that case both of these have limit equal to 1.

Example (section 3.3 exercise 50): Evaluate $\lim_{x \to 1} \dfrac{\sin(x-1)}{x^2+x-2}$

Solution: This limit is of the form $\frac{0}{0}$

$$= \lim_{x \to 1} \frac{\sin(x-1)}{(x-1)(x+2)}$$

$$= \lim_{x \to 1} \frac{\sin(x-1)}{(x-1)} \frac{1}{(x+2)}$$

$$= \left[\lim_{x \to 1} \frac{\sin(x-1)}{(x-1)}\right] \left[\lim_{x \to 1} \frac{1}{(x+2)}\right]$$

$$= \left[\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right] \left[\lim_{x \to 1} \frac{1}{(x+2)}\right]$$ change of variable $\theta = x - 1$

$$= (1)(\frac{1}{3})$$

$$= \frac{1}{3}$$

Exercise: Evaluate $\lim_{x \to -5} \dfrac{x^2 + 3x - 10}{\sin(x + 5)}$

Example : Evaluate $\lim\limits_{x\to 5} \dfrac{2\,\tan(x-5)}{x^2-6x+5}$

Solution: This is of the form $\frac{0}{0}$

$$= \lim_{x\to 5} \frac{2\,\tan(x-5)}{(x-5)(x-1)}$$

$$= \lim_{x\to 5} \frac{2\,\frac{\sin(x-5)}{\cos(x-5)}}{(x-5)(x-1)} = \lim_{x\to 5} \frac{2\,\sin(x-5)}{(x-5)(x-1)\cos(x-5)}$$

$$= \lim_{x\to 5} 2\frac{\sin(x-5)}{(x-5)}\frac{1}{(x-1)\cos(x-5)}$$

$$= 2\left[\lim_{x\to 5} \frac{\sin(x-5)}{(x-5)}\right]\left[\lim_{x\to 5} \frac{1}{(x-1)\cos(x-5)}\right]$$

$$= 2\left[\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right]\left[\lim_{x\to 5} \frac{1}{(x-1)\cos(x-5)}\right]$$ change of variable $\theta = x-5$

$$= 2(1)\frac{1}{(4)\cos(0)}$$

$$= 2(1)\frac{1}{(4)(1)} = \frac{1}{2}$$

Exercise: Evaluate $\lim_{x \to -3} \dfrac{x^2 + 4x + 3}{5\,\tan(x+3)}$

Example (section 3.3 exercise 43): Find the limit $\lim_{x\to 0} \dfrac{\sin 3x}{5x^3 - 4x}$

Solution:

This limit is of the form $\frac{0}{0}$. So we do this:

$$= \lim_{x\to 0} \left\{ \frac{\sin 3x}{3x} \; \frac{3x}{5x^3 - 4x} \right\}$$

$$= \left\{ \lim_{x\to 0} \frac{\sin 3x}{3x} \right\} \; \left\{ \lim_{x\to 0} \frac{3x}{5x^3 - 4x} \right\}$$

$$= \left\{ \lim_{\theta\to 0} \frac{\sin \theta}{\theta} \right\} \; \left\{ \lim_{x\to 0} \frac{3x}{5x^3 - 4x} \right\}$$ **change of variable $\theta = 3x$**

$$= (1) \; \left\{ \lim_{x\to 0} \frac{3x}{5x^3 - 4x} \right\}$$

$$= \left\{ \lim_{x\to 0} \frac{3}{5x^2 - 4} \right\}$$ **canceling out x**

$$= -\frac{3}{4}$$