Optimization Problems

sections 4.7

First-Derivative Test for Optimization Problems. Suppose that f' exists on interval I ; this interval can be of any type (closed , open, half open, bounded, unbounded, ...). Suppose that c is the only interior point of I with $f'(c) = 0$.

(i) If on both sides of c we have this situation:

		c	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

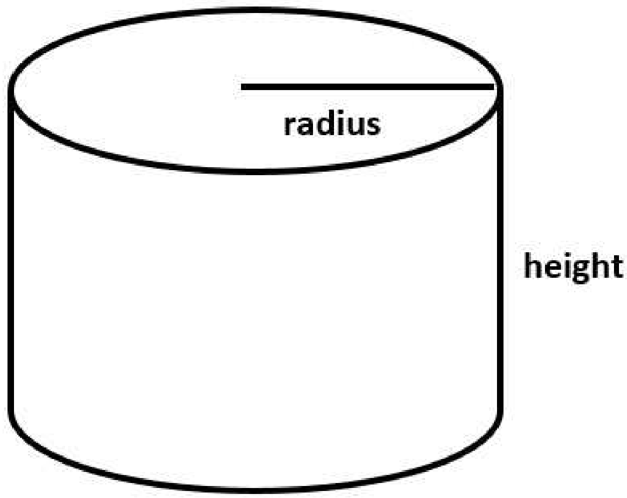
then f has an absolute minimum at c in the interval I.

(ii) If on both sides of c we have this situation:

		c	
f'	$+$		$-$
f	$\nearrow$		$\searrow$

then f has an absolute maximum at c in the interval I.

Example (from the textbook). A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions of the can that will minimize the cost of the metal to manufacture the can. Note that the surface area of a cylinder is $A = 2 \cdot \pi \cdot (\text{radius})^2 + 2 \cdot \pi \cdot \text{radius} \cdot \text{height}$ and the volume of it is $V = \pi \cdot (\text{radius})^2 \cdot \text{height}$



Solution: We denote the height by h and the radius by r.

Step 1: Create the target function and find its domain.

By considering cm as the unit of measurement for the length, we are given that :

$\pi r^2 h = 1 \text{ Liter} = 1000 \ \text{ cm}^3$

We want to minimize

$A = 2\pi r^2 + 2\pi rh$

We need to substitute for one of the variables. So, from $\pi r^2 h = 1000$ we write $h = \frac{1000}{\pi r^2}$ from which A changes to

$$A(r) = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

To have a cylinder, both the radius r and height h must be positive. So the first restriction on r is $r > 0$. Now can r be as large as it can, or there is an upper bound for r?. Well, in fact, if we choose any large number for r then from the equality $h = \dfrac{1000}{\pi r^2}$ we get a positive value for h which is an acceptable value for h and by having these values of r and h one can calculate A via $A = 2\pi r^2 + 2\pi r h$. In summary, the domain for r is $0 < r < \infty$.

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad 0 < r < \infty$$

Step 2: Form the table to find the minimum value .

$$A(r) = 2\pi r^2 + 2000\, r^{-1}$$

$$A'(r) = 4\pi r - 2000\, r^{-2} = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2} \qquad 0 < r < \infty$$

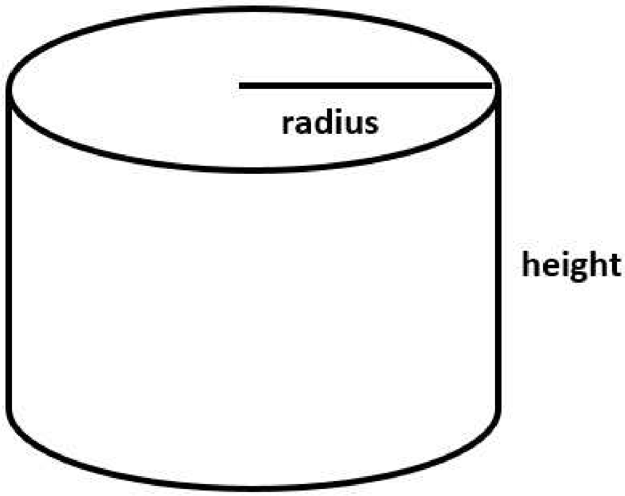
$$A'(r) = 0 \quad \Rightarrow \quad \pi r^3 - 500 = 0 \quad \Rightarrow \quad r^3 = \frac{500}{\pi} \quad \Rightarrow \quad r = \sqrt[3]{\frac{500}{\pi}}$$

	0		$\sqrt[3]{\frac{500}{\pi}}$	
$\pi r^3 - 500$		$-$	$\bullet$	$+$
A'		$-$		$+$
A		$\searrow$		$\nearrow$
			absolute min	

Step 3: Conclusion . The optimal dimensions are:

$$\begin{cases} r = \sqrt[3]{\frac{500}{\pi}} \\ \\ h = \frac{1000}{\pi \sqrt[3]{\left(\frac{500}{\pi}\right)^2}} \end{cases}$$

Example (from the textbook). A cylinder has a surface area of $150\,\pi$ m^2. Determine the dimensions of the cylinder that can maximize its volume. Note that the surface area of a cylinder is $A = 2 \cdot \pi \cdot (\text{radius})^2 + 2 \cdot \pi \cdot \text{radius} \cdot \text{height}$ and the volume of it is $V = \pi \cdot (\text{radius})^2 \cdot \text{height}$



Solution: We denote the height by h and the radius by r.

Step 1: Create the target function and find its domain.

We are given that :

$2\pi r^2 + 2\pi r h = 150\,\pi$ m^2

Find one of the variables in terms of the other one:

$$2\pi r^2 + 2\pi r h = 150\,\pi \quad \Rightarrow \quad r^2 + rh = 15 \quad \Rightarrow \quad h = \frac{75 - r^2}{r}$$

We want to maximize

$$V = \pi r^2 h = \pi r^2 \left(\frac{75 - r^2}{r}\right) = (\pi r)(75 - r^2) = 75\pi r - \pi r^3$$

So:

$V(r) = 75\pi r - \pi r^3$

We now need to specify the domain of this function. First of all both the radius and the height must be positive (otherwise it does not make sense to have a cylinder). Secondly, from $h = \dfrac{75 - r^2}{r}$ and from the fact that the height h must be positive, we are restricted to $75 - r^2 > 0$. So $r^2 < 75$ so $0 < r < \sqrt{75}$. Now we restate the function $V(r)$ and its domain:

$V(r) = 75\pi r - \pi r^3 \qquad 0 < r < \sqrt{75} = 5\sqrt{3}$

Step 2: Find the maximum value .

$V(r) = 75\pi r - \pi r^3 \qquad 0 < r < \sqrt{75}$

$V'(r) = 75\pi - 3\,\pi\, r^2 = 3\pi(25 - r^2) = 3\pi(5 - r)(5 + r)$

$V'(r) = 0 \quad \Rightarrow \quad 75\pi - 3\,\pi\, r^2 = 0 \quad \Rightarrow \quad r = 5$

The only term that affects the sign of $V'(r)$ is the term $5 - r$ because the other terms are positive.

	0	5	$5\sqrt{3}$
$5 - r$	$+$	$\bullet$	$-$
V'	$+$		$-$
V	$\nearrow$		$\searrow$

absolute max

Step 3: Conclusion . The optimal dimensions are:

$$\begin{cases} r = 5 \\ \\ h = \frac{75-r^2}{r} = \frac{75-25}{5} = 10 \end{cases}$$

Example (question 1 of the Lab 11). Find two numbers whose difference is 100 and whose product is a minimum.

Solution (using the first derivative test). Call them x and y. We are given $y - x = 100$, and we want to minimize the value $f = xy$. We first need to get rid of one of the variables and write the target function f as a function of one variable. For this we use $y = 100 + x$ and then substitute to get

$$f(x) = (100 + x)x = 100x + x^2 \qquad -\infty < x < \infty$$

Now:

$$f'(x) = 100 + 2x$$

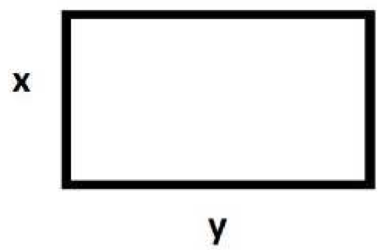
$$f'(x) = 0 \quad \Rightarrow \quad 100 + 2x = 0 \quad \Rightarrow \quad x = -50$$

		-50	
$100 + 2x$	$-$	$\bullet$	$+$
f'	$-$		$+$
f	$\searrow$		$\nearrow$
		absolute min	

So , the minimum happens when $x = -50$ and $y = 100 + x = 50$.

Conclusion: The optimal values are: $\begin{cases} x = -50 \\ y = 50 \end{cases}$

Example (question 2 of the Lab 11). Find the dimensions of a rectangle with area 900 cm^2 whose perimeter is as small as possible.



Solution:

Step 1: Create the target function and find its domain.

We are given that :

$xy = 900$

We want to minimize

$P = 2x + 2y$

We need to substitute for one of the variables. So, from $xy = 900$ we write $y = \frac{900}{x}$ from which P changes to

$$P(x) = 2x + 2\left(\frac{900}{x}\right) = 2x + \frac{1800}{x}$$

To have a rectangle we must have $0 < x$. For each positive value of x the positive value of y can be calculated via $y = \dfrac{900}{x}$ and then the perimeter can b calculated via $P = 2x + 2y$. Therefore there is no restriction on the positive values of x, so the domain for x is $0 < x < \infty$. So the function $P(x)$ has the domain:

$P(x) = 2x + \dfrac{1800}{x} \qquad 0 < x < \infty$

Step 2: Find the minimum value by forming tha table.

$P(x) = 2x + 1800\,x^{-1} \qquad 0 < x < \infty$

$P'(x) = 2 - 1800\,x^{-2} = 2 - \dfrac{1800}{x^2} = \dfrac{2x^2 - 1800}{x^2} \qquad 0 < x < \infty$

$P'(x) = 0 \quad \Rightarrow \quad 2x^2 - 1800 = 0 \quad \Rightarrow \quad 2x^2 = 1800 \quad \Rightarrow \quad x^2 = 900$

$\Rightarrow \quad x = \pm 30 \quad \Rightarrow \quad$ only $x = 30$ is acceptable to be in the domain

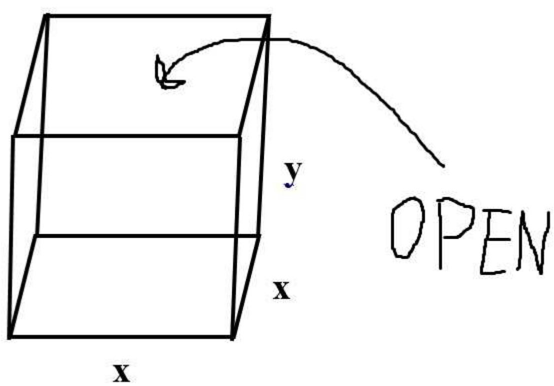
	0	30	
$2x^2 - 1800$	$-$	$\bullet$	$+$
f'	$-$		$+$
f	$\searrow$		$\nearrow$
		absolute min	

Step 3: Conclusion . The optimal dimensions are: $\begin{cases} x = 30 \\ y = \frac{900}{x} = \frac{900}{30} = 30 \end{cases}$

Example (Final Exam Fall 2012). You wish to construct a rectangular-open-top box with a square base. You have 48 square feet of plywood to make the sides and the bottom of the box. What should the dimensions of this box be so that its volume is maximized? (Ignore the thickness of the plywood in your calculations)

Solution (using the First Derivative Test).

Step 1: Draw the diagram and label variables.



Step 2: Create the target function and find its domain.

We are given that :

$x^2 + 4xy = 48$

We want to maximize

$V = x^2y$

We need to substitute for one of the variables. So, from $x^2 + 4xy = 48$ we write $y = \frac{48-x^2}{4x}$ from which V changes to

$V(x) = x^2 \left(\frac{48 - x^2}{4x} \right) = 12x - \frac{x^3}{4}$

To have a box we must have $x > 0$ and $y > 0$. From the equality $y = \frac{48 - x^2}{4x}$ it should be clear that to have $y > 0$ we must have $48 - x^2 > 0$ therefore the domain for x is $0 < x < \sqrt{48}$. So the domain of the function $V(x)$ is:

$$V(x) = 12x - \frac{x^3}{4} \qquad 0 < x < \sqrt{48}$$

Step 3: Find the maximum value .

$$V'(x) = 12 - \frac{3}{4}x^2 \qquad 0 < x < \sqrt{48}$$

$$V'(x) = 0 \quad \Rightarrow \quad 12 - \frac{3}{4}x^2 = 0 \quad \Rightarrow \quad x = \pm 4 \quad \Rightarrow \quad \text{only} \;\; x = 4 \text{ is acceptable}$$

To determine the sign of $12x - \frac{x^3}{4}$ before and after the point $x = 4$, i.e. on the intervals $(0\,,\,4)$ and $(4\,,\,\sqrt{48})$ substitute the points $x = 1$ and $x = 5$ to know the signs of V on those intervals.

	0	4	$\sqrt{48}$
$12x - \frac{x^3}{4}$	$+$	$\bullet$	$-$
V'	$+$		$-$
V	$\nearrow$		$\searrow$

absolute

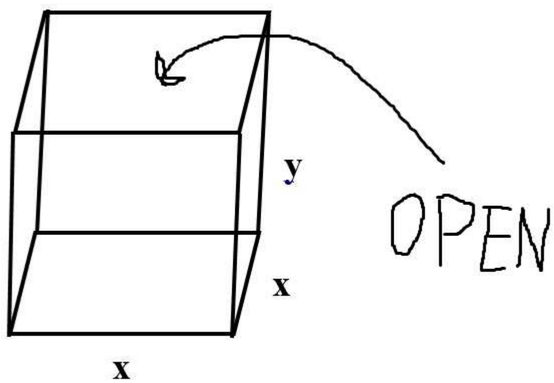
max

Step 4: Conclusion . The optimal dimensions are $x = 4$ and $y = \frac{48 - 4^2}{(4)(4)} = 2$.

Example (Final Exam Winter 2016). An open top box with a square base has a volume $V = 4m^3$. Find the dimensions of the box so that it uses the least amount of material.

Solution (using the First Derivative Test).

Step 1: Draw the diagram and label variables.



Step 2: Create the target function and find its domain.

We are given that :

$x^2 y = 4$

We want to minimize

$$A = 4xy + x^2 = 4x\left(\frac{4}{x^2} + x^2\right) = \frac{16}{x} + x^2$$

We need to substitute for one of the variables. So, from $x^2y = 4$ we write $y = \frac{4}{x^2}$ from which A changes to

$$A(x) = 4xy + x^2 = 4x\left(\frac{4}{x^2}\right) + x^2 = \frac{16}{x} + x^2$$

To have the box we must have $x > 0$. For any positive value of x the equality $y = \dfrac{4}{x^2}$ gives $y > 0$, so any positive value of x can be used , so the domain for x is $0 < x < \infty$.

$A(x) = 16x^{-1} + x^2 \qquad 0 < x < \infty$

Step 3: Find the maximum value .

$$A'(x) = -16x^{-2} + 2x = \frac{-16}{x^2} + 2x = \frac{-16 + 2x^3}{x^2} \qquad 0 < x < \infty$$

$$V'(x) = 0 \quad \Rightarrow \quad -16 + 2x^3 = 0 \quad \Rightarrow \quad x^3 = 8 \quad \Rightarrow \quad x = 2$$

The only factor that affects the sign of $V'(x)$ is the factor $-16 + 2x^3$. To determine the sign of $-16 + 2x^3$ before and after the point $x = 2$, i.e. on the intervals $(0\,,\,2)$ and $(2\,,\,\infty)$ substitute the points $x = 1$ and $x = 3$ to know the signs of V on those intervals.

	0		2	
$-16 + 2x^3$		$-$	$\bullet$	$+$
A'		$-$		$+$
A		$\searrow$		$\nearrow$
			absolute min	

Step 4: Conclusion . The optimal dimensions are $x = 2$ and $y = \dfrac{4}{x^2} = 1$.