Squeeze Theorem

Squeeze Theorem. Let lim denote any of the limits $\lim_{x\to a}$, $\lim_{x\to a^+}$, $\lim_{x\to a^-}$, $\lim_{x\to\infty}$, and $\lim_{x\to-\infty}$. Let for the points close to the point where the limit is being calculated at we have $f(x) \leq g(x) \leq h(x)$ (so for example if the limit $\lim_{x\to\infty}$ is being calculated then it is assumed that we have the inequalities $f(x) \leq g(x) \leq h(x)$ for all large x's). If under these assumptions we have $\lim f(x) = \lim h(x) = L$ then we have $\lim g(x) = L$

Note: There are two types of problems you are given in the exams and quizzes on the Squeeze Theorem. See the following examples:

Type I: the limit is zero

Example (section 2.3 exercise 39): Evaluate the limit $\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right)$

Solution: We first note that

$$\underbrace{x^4}_{\text{tend to zero}} \underbrace{\cos\left(\frac{2}{x}\right)}_{\text{bounded}}$$

so we answer the question in the following way:

$$-1 \le \cos \le 1 \quad \Rightarrow \quad -1 \le \cos\left(\frac{2}{x}\right) \le 1$$

multiply both sides with the positive number x^4 :

$$-x^4 \le x^4 \cos\left(\frac{2}{x}\right) \le x^4 \qquad (*)$$

But :

$$\lim_{x\to 0}(-x^4) = 0 \qquad \text{and} \qquad \lim_{x\to 0} x^4 = 0$$

So both sides of $(*)$ tend to zero, therefore by the Squeeze Theorem we will have:

$$\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$$

<u>**Example**</u> : Evaluate the limit $\lim\limits_{x \to 0} x^2 \sin\left(\frac{\pi}{x^3}\right)$

Solution: We first note that

$$\underbrace{x^2}_{\text{tend to zero}} \; \underbrace{\sin\left(\frac{\pi}{x^3}\right)}_{\text{bounded}}$$

so we answer the question in the following way:

$$-1 \leq \sin \leq 1 \qquad \Rightarrow \qquad -1 \leq \sin\left(\frac{\pi}{x^3}\right) \leq 1$$

multiply both sides with the positive number x^2 :

$$-x^2 \leq x^2 \sin\left(\frac{\pi}{x^3}\right) \leq x^2 \qquad (*)$$

But :

$$\lim_{x \to 0}(-x^2) = 0 \qquad \text{and} \qquad \lim_{x \to 0} x^2 = 0$$

So both sides of $(*)$ tend to zero, therefore by the Squeeze Theorem we will have:

$$\lim_{x \to 0} x^2 \sin\left(\frac{\pi}{x^3}\right) = 0$$

Example (Midterm Winter 2016): Evaluate the limit $\lim_{x\to 1}(x-1)^2\cos\left(\frac{1}{x-1}\right)$

Solution: We first note that

$$\underbrace{(x-1)^2}_{\text{tend to zero}}\ \underbrace{\cos\left(\frac{1}{x-1}\right)}_{\text{bounded}}$$

so we answer the question in the following way:

$$-1 \le \cos \le 1 \quad \Rightarrow \quad -1 \le \cos\left(\frac{1}{x-1}\right) \le 1$$

multiply both sides with the positive number $(x-1)^2$:

$$-(x-1)^2 \le (x-1)^2\cos\left(\frac{1}{x-1}\right) \le (x-1)^2 \qquad (*)$$

But :

$$\lim_{x\to 1}(-(x-1)^2) = 0 \qquad \text{and} \qquad \lim_{x\to 1}(x-1)^2 = 0$$

So both sides of $(*)$ tend to zero, therefore by the Squeeze Theorem we will have:

$$\lim_{x\to 1}(x-1)^2\cos\left(\frac{1}{x-1}\right) = 0$$

Example (Midterm Fall 2015): Evaluate the limit $\lim_{x\to 2^+} \sqrt{x-2}\,\cos\left(\frac{1}{x-2}\right)$

Solution: We first note that

$$\underbrace{\sqrt{x-2}}_{\text{tend to zero}}\ \underbrace{\cos\left(\frac{1}{x-2}\right)}_{\text{bounded}}$$

so we answer the question in the following way:

$$-1 \le \cos \le 1 \qquad \Rightarrow \qquad -1 \le \cos\left(\frac{1}{x-2}\right) \le 1$$

multiply both sides with the positive number $\sqrt{x-2}$:

$$-\sqrt{x-2} \le \sqrt{x-2}\,\cos\left(\frac{1}{x-2}\right) \le \sqrt{x-2} \qquad\qquad (*)$$

But :

$$\lim_{x\to 2^+} (-\sqrt{x-2}) = 0 \qquad \text{and} \qquad \lim_{x\to 2^+} \sqrt{x-2} = 0$$

So both sides of $(*)$ tend to zero, therefore by the Squeeze Theorem we will have:

$$\lim_{x\to 2^+} \sqrt{x-2}\,\cos\left(\frac{1}{x-2}\right) = 0$$

Type II: the limit is nonzero

Example (section 2.3 exercise 37): If $4x - 9 \leq f(x) \leq x^2 - 4x + 7$ for $x \geq 0$, find $\lim_{x \to 4} f(x)$

Solution:

$\lim_{x \to 4} (4x - 9) = 16 - 9 = 7$ **(this gets some marks)**

$\lim_{x \to 4} (x^2 - 4x + 7) = 16 - 16 + 7 = 7$ **(this gets some marks)**

Therefore by the Squeeze Theorem we will have $\lim_{x \to 4} f(x) = 7$ **(this gets some marks)**

Example : A function $f(x)$ satisfies the inequalities $\dfrac{2\sin\left(\frac{\pi}{2}x\right)}{x-3} < f(x) < -\,e^{x-1}$ when $x \to 1^-$. Find $\lim\limits_{x\to 1^-} f(x)$.

Solution:

$$\lim_{x\to 1^-} \frac{2\sin\left(\frac{\pi}{2}x\right)}{x-3} = \frac{2\sin\left(\frac{\pi}{2}\right)}{-2} = \frac{2}{-2} = -1$$ **(this gets some marks)**

$$\lim_{x\to 1^-} \left(-\,e^{x-1}\right) = -e^0 = -1$$ **(this gets some marks)**

Therefore by the Squeeze Theorem we will have $\lim\limits_{x\to 1^-} f(x) = -1$ **(this gets some marks)**

Exercise : A function $g(x)$ satisfies the inequalities $2x < g(x) < x^4 - x^2 + 2$ when $x \to 2^+$.

Find $\lim_{x \to 2^+} g(x)$.