One Sided Derivatives

The left-hand derivative and right-hand derivative are define by:

$$f'_{-}(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

$$f'_{+}(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

Theorem : For $f'(a)$ to exist it is necessary and sufficient that these conditions are met:

a) both $f'_{-}(a)$ and $f'_{+}(a)$ exist

b) $f'_{-}(a) = f'_{+}(a)$

Furthermore , if these conditions are met, then the derivative $f'(a)$ equals the common value of $f'_{-}(a)$ and $f'_{+}(a)$:

$$f'(a) = f'_{-}(a) = f'_{+}(a)$$

Example : Is the function $f(x) = |x+7|$ differentiable at $x = -7$?

Solution:

We have $f(-7) = 0$. So:

$$f'_-(-7) = \lim_{h\to 0^-} \frac{f(-7+h)-f(-7)}{h}$$

$$= \lim_{h\to 0^-} \frac{|(-7+h)+7|-0}{h}$$

$$= \lim_{h\to 0^-} \frac{|h|}{h}$$

as $h < 0$ in this case

$$= \lim_{h\to 0^-} \frac{-h}{h} = -1$$

On the other hand:

$$f'_+(-7) = \lim_{h\to 0^+} \frac{f(-7+h)-f(-7)}{h}$$

$$= \lim_{h\to 0^+} \frac{|(-7+h)+7|-0}{h}$$

$$= \lim_{h\to 0^+} \frac{|h|}{h}$$

as $h > 0$ in this case

$$= \lim_{h\to 0^+} \frac{h}{h} = 1$$

Then:

$f'_-(-7) \neq f'_+(-7) \quad \Longrightarrow \quad$ it is not differentiable at $x = -7$

Example : Is the function $f(x) = |x-3| + x^2$ differentiable at $x = 3$.

Solution:

$$\begin{aligned}
f'_{-}(3) &= \lim_{h\to 0^-} \tfrac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-} \tfrac{\left\{|h|+(3+h)^2\right\}-\{9\}}{h} \\
&= \lim_{h\to 0^-} \tfrac{\left\{|h|+(9+6h+h^2)\right\}-\{9\}}{h} = \lim_{h\to 0^-} \tfrac{|h|+6h+h^2}{h} \\
&= \lim_{h\to 0^-} \tfrac{(-h)+6h+h^2}{h} = \lim_{h\to 0^-} \tfrac{5h+h^2}{h} \\
&= \lim_{h\to 0^-} (5+h) = 5
\end{aligned}$$

On the other hand:

$$\begin{aligned}
f'_{+}(3) &= \lim_{h\to 0^+} \tfrac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+} \tfrac{\left\{|h|+(3+h)^2\right\}-\{9\}}{h} \\
&= \lim_{h\to 0^+} \tfrac{\left\{|h|+(9+6h+h^2)\right\}-\{9\}}{h} = \lim_{h\to 0^+} \tfrac{|h|+6h+h^2}{h} \\
&= \lim_{h\to 0^+} \tfrac{(h)+6h+h^2}{h} = \lim_{h\to 0^+} \tfrac{7h+h^2}{h} \\
&= \lim_{h\to 0^+} (7+h) = 7
\end{aligned}$$

Then:

$f'_{-}(3) \neq f'_{+}(3) \quad \Longrightarrow \quad$ it is not differentiable at $x = 3$