Squeeze Theorem

Squeeze Theorem. Let lim denote any of the limits $\lim_{x\to a}$, $\lim_{x\to a^+}$, $\lim_{x\to a^-}$, $\lim_{x\to\infty}$, and $\lim_{x\to-\infty}$. Let for the points close to the point where the limit is being calculated at we have $f(x) \le g(x) \le h(x)$ (so for example if the limit $\lim_{x\to\infty}$ is being calculated then it is assumed that we have the inequalities $f(x) \le g(x) \le h(x)$ for all large x's). If under these assumptions we have $\lim f(x) = \lim h(x) = L$ then we have $\lim g(x) = L$

Note: Before attempting the next examples let us recall from the file of "Exponential Functions" that $e^{-\infty} = 0$.

Example: Use the Squeeze Theorem to evaluate the limit $\displaystyle\lim_{x\to 0^-} e^{\frac{5}{7x}} \cos\left(\frac{2e}{x}\right)$

Solution: The function cosine is always bounded by -1 and 1:

$$-1 \le \cos\left(\frac{2e}{x}\right) \le 1$$

On the other hand, $e^{\frac{5}{7x}} > 0$ as the exponential function returns positive values. Then we multiply the inequality $-1 \le \cos\left(\frac{2e}{x}\right) \le 1$ by the positive value $e^{\frac{5}{7x}} > 0$ to get:

$$-e^{\frac{5}{7x}} \le e^{\frac{5}{7x}} \cos\left(\frac{2e}{x}\right) \le e^{\frac{5}{7x}}$$

Then:

$$\lim_{x\to 0^-}\left\{-e^{\frac{5}{7x}}\right\} \le \lim_{x\to 0^-}\left\{e^{\frac{5}{7x}} \cos\left(\frac{2e}{x}\right)\right\} \le \lim_{x\to 0^-}\left\{e^{\frac{5}{7x}}\right\}$$

$$-e^{\left(\frac{5}{0^-}\right)} \le \lim_{x\to 0^-}\left\{e^{\frac{5}{7x}} \cos\left(\frac{2e}{x}\right)\right\} \le e^{\left(\frac{5}{0^-}\right)}$$

$$-e^{-\infty} \le \lim_{x\to 0^-}\left\{e^{\frac{5}{7x}} \cos\left(\frac{2e}{x}\right)\right\} \le e^{-\infty}$$

$$0 \leq \lim_{x \to 0^-} \left\{ e^{\frac{5}{7x}} \cos\left(\frac{2e}{x}\right) \right\} \leq 0$$

$$\Rightarrow \quad \lim_{x \to 0^-} \left\{ e^{\frac{5}{7x}} \cos\left(\frac{2e}{x}\right) \right\} = 0$$

Example: Use the Squeeze Theorem to evaluate the limit $\lim_{x \to 0^+} 4\,e^{-\frac{2}{x^2}} \sin\left(\frac{\pi}{2x^2}\right)$

Solution: The function sine is always bounded by -1 and 1:

$$-1 \leq \sin\left(\frac{\pi}{2x^2}\right) \leq 1$$

On the other hand, $4\,e^{-\frac{2}{x^2}} > 0$ as the exponential function returns positive values. Then we multiply the inequality $-1 \leq \sin\left(\frac{\pi}{2x^2}\right) \leq 1$ by the positive value $4\,e^{-\frac{2}{x^2}} > 0$ to get:

$$-4\,e^{-\frac{2}{x^2}} \leq 4\,e^{-\frac{2}{x^2}} \sin\left(\frac{\pi}{2x^2}\right) \leq 4\,e^{-\frac{2}{x^2}}$$

Then:

$$\lim_{x \to 0^+} \left\{ -4\,e^{-\frac{2}{x^2}} \right\} \leq \lim_{x \to 0^+} \left\{ 4\,e^{-\frac{2}{x^2}} \sin\left(\frac{\pi}{2x^2}\right) \right\} \leq \lim_{x \to 0^+} \left\{ 4\,e^{-\frac{2}{x^2}} \right\}$$

$$\lim_{x \to 0^+} \left\{ -4\,e^{-\frac{2}{0^+}} \right\} \leq \lim_{x \to 0^+} \left\{ 4\,e^{-\frac{2}{x^2}} \sin\left(\frac{\pi}{2x^2}\right) \right\} \leq \lim_{x \to 0^+} \left\{ 4\,e^{-\frac{2}{0^+}} \right\}$$

$$-4\,e^{-\infty} \leq \lim_{x \to 0^+} \left\{ 4\,e^{-\frac{2}{x^2}} \sin\left(\frac{\pi}{2x^2}\right) \right\} \leq 4\,e^{-\infty}$$

$$0 \leq \lim_{x \to 0^+} \left\{ 4\,e^{-\frac{2}{x^2}} \sin\left(\frac{\pi}{2x^2}\right) \right\} \leq 0$$

$$\Rightarrow \quad \lim_{x \to 0^+} \left\{ 4\,e^{-\frac{2}{x^2}} \sin\left(\frac{\pi}{2x^2}\right) \right\} = 0$$