Finding inverse using LU decomposition (section 4.6.1)

Example. Find the inverse of the following upper triangular matrix:

$$U = \begin{bmatrix} 2 & 4 & 6 \\ 0 & -1 & -8 \\ 0 & 0 & 96 \end{bmatrix}$$

Solution.

$[U|I] =$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & -1 & -8 & 0 & 1 & 0 \\ 0 & 0 & 96 & 0 & 0 & 1 \end{array}\right] \left(\frac{1}{96}R_3\right) \left[\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & -1 & -8 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{96} \end{array}\right] \begin{array}{c} -6R_3 \overset{\text{add to}}{\longrightarrow} R_1 \\ 8R_3 \overset{\text{add to}}{\longrightarrow} R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 0 & 1 & 0 & -\frac{1}{16} \\ 0 & -1 & 0 & 0 & 1 & \frac{1}{12} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{96} \end{array}\right] (-R_2) \left[\begin{array}{ccc|ccc} 2 & 4 & 0 & 1 & 0 & -\frac{1}{16} \\ 0 & 1 & 0 & 0 & -1 & -\frac{1}{12} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{96} \end{array}\right] - 4R_2 \overset{\text{add to}}{\longrightarrow} R_1$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 4 & \frac{13}{48} \\ 0 & 1 & 0 & 0 & -1 & -\frac{1}{12} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{96} \end{array}\right] (\frac{1}{2}R_1) \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 2 & \frac{13}{96} \\ 0 & 1 & 0 & 0 & -1 & -\frac{1}{12} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{96} \end{array}\right] = [\,\text{identity}\,|\,U^{-1}]$$

and as soon as we get the identity matrix on the left-hand part, then the matrix on the right-hand part is the inverse to the upper triangular matrix:

$$U^{-1} = \begin{bmatrix} \frac{1}{2} & 2 & \frac{13}{96} \\ 0 & -1 & -\frac{1}{12} \\ 0 & 0 & \frac{1}{96} \end{bmatrix}$$

Example. Find the inverse of the following lower triangular matrix:

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 8 & -7 & 0 \\ 4 & 9 & -27 \end{bmatrix}$$

Solution.

$[L|I] =$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 8 & -7 & 0 & 0 & 1 & 0 \\ 4 & 9 & -27 & 0 & 0 & 1 \end{array}\right] \left(\frac{1}{2}R_1\right) \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 8 & -7 & 0 & 0 & 1 & 0 \\ 4 & 9 & -27 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -8R_1 \xrightarrow{\text{add to}} R_2 \\ -4R_1 \xrightarrow{\text{add to}} R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -7 & 0 & -4 & 1 & 0 \\ 0 & 9 & -27 & -2 & 0 & 1 \end{array}\right] \left(-\frac{1}{7}R_2\right) \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{4}{7} & -\frac{1}{7} & 0 \\ 0 & 9 & -27 & -2 & 0 & 1 \end{array}\right] -\tfrac{1}{9}R_2 \xrightarrow{\text{add to}} R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{4}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & -27 & -\frac{50}{7} & \frac{9}{7} & 1 \end{array}\right] \left(-\frac{1}{27}R_3\right) \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{4}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{50}{189} & -\frac{1}{21} & -\frac{1}{27} \end{array}\right] = [\,\text{identity}\,|\,L^{-1}]$$

So:

$$L^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{4}{7} & -\frac{1}{7} & 0 \\ \frac{50}{189} & -\frac{1}{21} & -\frac{1}{27} \end{bmatrix}$$

Note. If the upper triangular matrix or lower triangular matrix has 1 all over the main diagonal, then there is no need to apply the row operations to get the inverse, you only need to change the signs of the off-diagonal elements.

Example.

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 3 & 14 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\mathbf{\frac{3}{2}} & 1 & 0 \\ \mathbf{-3} & \mathbf{-14} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{4}{3} & -\frac{5}{6} \\ 0 & 1 & \frac{3}{7} \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\mathbf{\frac{4}{3}} & \mathbf{\frac{5}{6}} \\ 0 & 1 & -\mathbf{\frac{3}{7}} \\ 0 & 0 & 1 \end{bmatrix}$$

Note. The inverse operator has the following property:

$$A = BC \qquad \Rightarrow \qquad A^{-1} = C^{-1}B^{-1}$$

Example. Find the inverse of the matrix A that has the LU decomposition:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 3 & 14 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & -1 & -8 \\ 0 & 0 & 96 \end{bmatrix}$$

Solution. Using our findings in the first example , we can write:

$$A^{-1} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & -1 & -8 \\ 0 & 0 & 96 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 3 & 14 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 2 & \frac{13}{96} \\ 0 & -1 & -\frac{1}{12} \\ 0 & 0 & \frac{1}{96} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ -3 & -14 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{93}{32} & \frac{5}{48} & \frac{13}{96} \\ \frac{7}{4} & \frac{1}{6} & -\frac{1}{12} \\ -\frac{1}{32} & -\frac{7}{48} & \frac{1}{96} \end{bmatrix}$$

So here is twp-step procedure to find the inverse of a matrix A:

Step 1.. Find the LU decomposition $A = LU$ (Gaussian form or the Crout form whichever you are told to find)

Step 2.. Find the inverse of $A^{-1} = U^{-1}L^{-1}$ by inverting the matrices U and L.