Questions for Lab Session 4
(Tuesday Feb. 2)

1. Solve exercise 6.2 of the textbook.

The following data is given:

x	-7	-4	-1	0	2	5	7
y	20	14	5	3	-2	-10	-15

(a) Use linear least-squares regression to determine the coefficients m and b in the function $y = mx + b$ that best fit the data.

(b) Use Eq. (6.5) to determine the overall error.

Solution.

To comply with the model, the first column of X should be the input values x and the second column should be a column of ones.

$$X = \begin{bmatrix} -7 & 1 \\ -4 & 1 \\ -1 & 1 \\ 0 & 1 \\ 2 & 1 \\ 5 & 1 \\ 7 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 20 \\ 14 \\ 5 \\ 3 \\ -2 \\ -10 \\ -15 \end{bmatrix}$$

Then

$$X' * X = \begin{bmatrix} 144 & 2 \\ 2 & 7 \end{bmatrix} \qquad X' * Y = \begin{bmatrix} -360 \\ 15 \end{bmatrix}$$

$$(X' * X)^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 0.0070 & -0.0020 \\ -0.0020 & 0.1434 \end{bmatrix}$$

$$(X' * X)^{-1} X' * Y = \begin{bmatrix} 0.0070 & -0.0020 \\ -0.0020 & 0.1434 \end{bmatrix} \begin{bmatrix} 15 \\ -360 \end{bmatrix} = \begin{bmatrix} -2.5398 \\ 2.8685 \end{bmatrix} = \begin{bmatrix} m \\ b \end{bmatrix}$$

The polynomial: $p(x) = -2.5398\, x + 2.8685$

The values of $p(x)$ at the x values -7 , -4 , -1 , 0 , 2 , 5 , 7 are:

$$y_{\text{fitted}} = \begin{bmatrix} 20.6474 \\ 13.0279 \\ 5.4084 \\ 2.8685 \\ -2.2112 \\ -9.8307 \\ -14.9104 \end{bmatrix}$$

The difference between this vector and the vector y is

$$y - y_{\text{fitted}} = \begin{bmatrix} 20 \\ 14 \\ 5 \\ 3 \\ -2 \\ -10 \\ -15 \end{bmatrix} - \begin{bmatrix} 20.6474 \\ 13.0279 \\ 5.4084 \\ 2.8685 \\ -2.2112 \\ -9.8307 \\ -14.9104 \end{bmatrix} = \begin{bmatrix} -0.6474 \\ 0.9721 \\ -0.4084 \\ 0.1315 \\ 0.2112 \\ -0.1693 \\ -0.0896 \end{bmatrix}$$

Squaring these values gives us:

$$(-0.6474)^2+(0.9721)^2+(-0.4084)^2+(0.1315)^2+(0.2112)^2+(-0.1693)^2+(-0.0896)^2 = 1.6295$$

We are asked to use linear least-squares regression to determine the coefficients m and b in the function that best fit the data. The Matlab function `polyfit` does the job for us. So here is the code:

```
x = [-7  -4  -1  0  2  5  7] ;
y = [20  14  5  3  -2  -10  -15] ;
p = polyfit(x, y, 1) ;
a1 = p(1) ;
a0 = p(2) ;
y_fitted = a1 * x + a0 ;
norm(y - y_fitted) ^ 2
```

By running this m-file one gets the same overall error 1.6295 we found above.

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2. Solve exercise 6.6 of the textbook.

The following data is given:

x	**1**	**2**	**3**	**5**	**8**
y	**0.8**	**1.9**	**2.2**	**3**	**3.5**

Determine the coefficients m and b in the function $y = \left[m\sqrt{x} + b\right]^{\frac{1}{2}}$ that best fit the data. Write the equation in a linear form , and use linear least-squares regression to determine the values of the coefficients.

Solution.

We first write the model in the form

$$y^2 = m\sqrt{x} + b$$

which is now a linear model. To comply with this model, we put the squares of the output values y in the column Y. Further, the first column of X must be the values $\sqrt{x}$ and the second column of X must be a column of ones.

$$X = \begin{bmatrix} 1 & 1 \\ 1.4142 & 1 \\ 1.7321 & 1 \\ 2.2361 & 1 \\ 2.8284 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 0.64 \\ 3.61 \\ 4.84 \\ 9 \\ 12.25 \end{bmatrix}$$

Then

$$X' * X = \begin{bmatrix} 19 & 9.2108 \\ 9.2108 & 5 \end{bmatrix} \qquad X' * Y = \begin{bmatrix} 68.9013 \\ 30.3400 \end{bmatrix}$$

$$(X' * X)^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 0.4920 & -0.9064 \\ -0.9064 & 1.8697 \end{bmatrix}$$

$$(X' * X)^{-1} X' * Y = \begin{bmatrix} 0.4920 & -0.9064 \\ -0.9064 & 1.8697 \end{bmatrix} \begin{bmatrix} 68.9013 \\ 30.3400 \end{bmatrix} = \begin{bmatrix} 6.4015 \\ -5.7246 \end{bmatrix} = \begin{bmatrix} m \\ b \end{bmatrix}$$

The required function is: $f(x) = \left[6.4015\,\sqrt{x} + -5.7246\right]^{\frac{1}{2}}$

The values of $f(x)$ at the x values are:

$$y_{\text{fitted}} = \begin{bmatrix} 0.8228 \\ 1.8244 \\ 2.3159 \\ 2.9308 \\ 3.5188 \end{bmatrix}$$

The difference between this vector and the vector y is

$$y - y_{\text{fitted}} = \begin{bmatrix} 0.8 \\ 1.9 \\ 2.2 \\ 3 \\ 3.5 \end{bmatrix} - \begin{bmatrix} 0.8228 \\ 1.8244 \\ 2.3159 \\ 2.9308 \\ 3.5188 \end{bmatrix} = \begin{bmatrix} -0.0228 \\ 0.0756 \\ -0.1159 \\ 0.0692 \\ -0.0188 \end{bmatrix}$$

Squaring these values gives us:

$$(-0.0228)^2 + (0.0756)^2 + (-0.1159)^2 + (0.0692)^2 + (-0.0188)^2 = 0.0248$$

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3. Solve exercise 6.9 of the textbook.

In an electrophoretic fiber-making process, the diameter of the fiber , d , is related to the current flow, I. The following are measure during production:

I	**300**	**300**	**350**	**400**	**400**	**500**	**500**	**650**	**650**
d	**22**	**26**	**27**	**30**	**34**	**33**	**33.5**	**37**	**42**

The relation between the current and the diameter can be modeled with an equation of the form $d = a + b\sqrt{I}$. Use the data to determine the constants a and b that best fit the data.

Solution.

To comply with the model, we must put a column of ones as the first column of X and must put the values $\sqrt{I}$ as the second column of it. The values of d form the column vector Y.

$$X = \begin{bmatrix} 1 & 17.3205 \\ 1 & 17.3205 \\ 1 & 18.7083 \\ 1 & 20 \\ 1 & 20 \\ 1 & 22.3607 \\ 1 & 22.3607 \\ 1 & 25.4951 \\ 1 & 25.4951 \end{bmatrix} \qquad Y = \begin{bmatrix} 22 \\ 26 \\ 27 \\ 30 \\ 34 \\ 33 \\ 33.5 \\ 37 \\ 42 \end{bmatrix}$$

Then

$$X' * X = \begin{bmatrix} 9 & 189.0609 \\ 189.0609 & 4050 \end{bmatrix} \qquad X' * Y = \begin{bmatrix} 284.5000 \\ 6117.6060 \end{bmatrix}$$

$$(X' * X)^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 5.7366 & -0.2678 \\ -0.2678 & 0.0127 \end{bmatrix}$$

$$(X' * X)^{-1} X' * Y = \begin{bmatrix} 5.7366 & -0.2678 \\ -0.2678 & 0.0127 \end{bmatrix} \begin{bmatrix} 284.5000 \\ 6117.6060 \end{bmatrix} = \begin{bmatrix} -6.1967 \\ 1.7998 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

The required function is: $\quad f(x) = -6.1967 + 1.7998\sqrt{I}$

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4. Solve exercise 6.10 of the textbook.

Determine the coefficients of the polynomial $y = a_2\,x^2 + a_1 x + a_0$ that best the data given in problem 6.5:

x	**-2**	**-1**	**0**	**1**	**2**
y	**1.5**	**3.2**	**4.5**	**3.4**	**2**

Solution.

To comply with the model, we must put the squares x^2 of the input values as the first column of X, the input values x as the second column of X, and finally a column of ones as the last column of X.

$$X = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 1.5 \\ 3.2 \\ 4.5 \\ 3.4 \\ 2 \end{bmatrix}$$

Then

$$X' * X = \begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 5 \end{bmatrix} \qquad X' * Y = \begin{bmatrix} 20.6 \\ 1.2 \\ 14.6 \end{bmatrix}$$

Now use the Matlab function inv() to invert the matrix $X' * X$ by commanding Matlab with `inv(X' * X)` :

$$(X' * X)^{-1} = \begin{bmatrix} 0.0714 & 0 & -0.1429 \\ 0 & 0.1 & 0 \\ -0.1429 & 0 & 0.4857 \end{bmatrix}$$

$$(X' * X)^{-1} X' * Y = \begin{bmatrix} -0.6143 \\ 0.1200 \\ 4.1486 \end{bmatrix} = \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

The required polynomial is: $p(x) = -0.6143\,x^2 + 0.1200\,x + 4.1486$

5. Solve exercise 6.20 of the textbook.

6. Solve exercise 8.2 of the textbook.

7. Solve exercise 6.31 of the textbook.