1 Multistep Methods (section 10.6)

The methods we have been working with all use the approximate value y_i solely to get the next approximate value y_{i+1}. Here we want to describe some methods that incorporate some of the previous approximate values $\left\{y_{i-1}, y_{i-2}, \ldots\right\}$ too. These methods are called multistep methods. Some of them are explicit methods and some are implicit methods.

2 Adams-Bashforth Methods (section 10.6.1)

The Adams-Bashforth Method is an explicit multistep method. The first of these methods we describe is the called Third-Order Adams-Bashforth Method. We justify this method through the following discussion:

Consider the differential equation

$$y'(x) = f(x, y(x))$$

Take three nodes $\{x_{i-2}\ ,\ x_{i-1}\ ,\ x_i\}$ with step size h.

Step 1. We first find the coefficients A, B, and C such that the approximating formula

$$\int_0^h \varphi(x)dx \approx A\varphi(-2h) + B\varphi(-h) + C\varphi(0)$$

is exact for all polynomials of degree at most 2.

$$\begin{cases} \varphi(x) = 1 \quad \Rightarrow \quad A + B + C = \int_0^h (1)dx = h & (1) \\ \\ \varphi(x) = x \quad \Rightarrow \quad A(-2h) + B(-h) = \int_0^h x dx = \frac{1}{2}h^2 & (2) \\ \\ \varphi(x) = x^2 \quad \Rightarrow \quad A(-2h)^2 + B(-h)^2 = \int_0^h x^2 dx = \frac{1}{3}h^3 & (3) \end{cases}$$

Drop a factor of h from equation (2), and drop a factor of h from equation (3), to simplify to:

$$\begin{cases} \varphi(x) = 1 \quad \Rightarrow \quad A + B + C = \int_0^h (1)dx = h & (1) \\ \\ f(x) = x \quad \Rightarrow \quad -2A - B = \frac{1}{2}h & (2') \\ \\ f(x) = x^2 \quad \Rightarrow \quad 4A + B = \frac{1}{3}h & (3') \end{cases}$$

By adding equations $(2')$ and $(3')$ together, we get:

$$2A = \frac{5}{6}h \quad \Rightarrow \quad \boxed{A = \tfrac{5}{12}h}$$

Putting this into $(3')$ gives us $\boxed{B = \tfrac{-16}{12}h}$

Then from (1) we get:

$$C = h - A - B \quad \Rightarrow \quad \boxed{C = \tfrac{23}{12}h}$$

So the approximating formula is:

$$\int_0^h \varphi(x)dx \approx \frac{5}{12}h\,\varphi(-2h) - \frac{16}{12}h\,\varphi(-h) + \frac{23}{12}h\,\varphi(0) = \frac{h}{12}\Big[5\,\varphi(-2h) - 16\,\varphi(-h) + 23\,\varphi(0)\Big]$$

Step 2. Now let $y' = f(x, y)$ be the differential equation under consideration. Then

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} y'(x)dx$$

$$= \int_0^h y'(t + x_i)\,dt \qquad \text{change of variable} \begin{cases} t = x - x_i \\ dt = dx \end{cases}$$

$$\approx \frac{h}{12}\Big[5\,y'(-2h + x_i) - 16\,y'(-h + x_i) + 23\,y'(0 + x_i)\Big]$$

$$= \frac{h}{12}\Big[5\,y'(x_{i-2}) - 16\,y'(x_{i-1}) + 23\,y'(x_i)\Big]$$

$$= \frac{h}{12}\Big[5\,f(x_{i-2}\,,\,y(x_{i-2})) - 16\,f(x_{i-1}\,,\,y(x_{i-1})) + 23\,f(x_i\,,\,y(x_i))\Big]$$

substitute approximate values of y for real values of y as the real values are not accessible

$$\approx \frac{h}{12}\Big[5\,f(x_{i-2}\,,\,y_{i-2}) - 16\,f(x_{i-1}\,,\,y_{i-1}) + 23\,f(x_i\,,\,y_i)\Big]$$

In this way we get the **Third-Order Adams-Bashforth Method**

$$y_{i+1} = y_i + \tfrac{h}{12}\Big[23\,f(x_i\,,\,y_i) - 16\,f(x_{i-1}\,,\,y_{i-1}) + 5\,f(x_{i-2}\,,\,y_{i-2})\Big]$$

Note. If we started by finding the coefficients A and B in the approximating formula

$$\int_0^h \varphi(x)dx \approx A\varphi(-h) + B\varphi(0)$$

such that the formula is exact for polynomials of degree less than or equal to 1, then we would get the so-called **Second-Order Adams-Bashforth Method** :

$$y_{i+1} = y_i + \tfrac{h}{2}\Big[3\,f(x_i\,,\,y_i) - f(x_{i-1}\,,\,y_{i-1})\Big]$$

Note. If we started by finding the coefficients A and B in the approximating formula

$$\int_0^h \varphi(x)dx \approx A\varphi(-3h) + B\varphi(-2h) + C\varphi(-h) + D\varphi(0)$$

such that the formula is exact for polynomials of degree less than or equal to 3, then we would get the so-called **Fourth-Order Adams-Bashforth Method** :

$$y_{i+1} = y_i + \tfrac{h}{24}\Big[55\,f(x_i\,,\,y_i) - 59\,f(x_{i-1}\,,\,y_{i-1}) + 37\,f(x_{i-2}\,,\,y_{i-2}) - 9f(x_{i-3}\,,\,y_{i-3})\Big]$$

In the textbook the slopes

$$f(x_i\,,y_i) \quad , \quad f(x_{i-1}\,,y_{i-1}) \quad , \quad f(x_{i-2}\,,y_{i-2}) \quad , \quad f(x_{i-3}\,,y_{i-3})$$

are denoted by

$$f_i \quad , \quad f_{i-1} \quad , \quad f_{i-2} \quad , \quad f_{i-3}$$

respectively, and therefore the above identity is written in the form:

$$y_{i+1} = y_i + \tfrac{h}{24}\Big[55\,f_i - 59\,f_{i-1} + 37\,f_{i-2} - 9f_{i-3}\Big]$$

Note. To be able to use the Fourth-Order Adams-Bashforth Method, one needs the initial values $\{y_1, y_2, y_3, y_4\}$ to start with. They can be found using the RK2, or RK4 method, or any other single-step method.

Example. Use the Adams-Bashforth to approximate $y(0.8)$ for the function satisfying the following differential equation:

$$y' = 1 + y^2 \qquad y = 0 \qquad x = 0$$

Take $h = 0.2$

Solution. Applying the RK4 method, we get the following values:

$$\begin{cases} y_1 = 0 \\ y_2 = 0.2027 \\ y_3 = 0.4228 \\ y_4 = 0.6841 \end{cases}$$

Now we use the Fourth Order Adams-Bashforth to get :

$$y_5 = y_4 + \frac{h}{24}\left[55\, f(x_i, y_i) - 59\, f(x_{i-1}, y_{i-1}) + 37\, f(x_{i-2}, y_{i-2}) - 9f(x_{i-3}, y_{i-3})\right]$$

$$= 0.6841 + \frac{0.2}{24}\left[55(1+0.6841^2) - 59(1+0.4228^2) + 37(1+0.2027^2) - 9(1+0^2)\right] = 1.0233$$

3 Adams-Moulton Methods (section 10.6.2)

Step 1. We first find the coefficients A, B, and C such that the approximating formula

$$\int_0^h \varphi(x)dx \approx A\varphi(-h) + B\varphi(0) + C\varphi(h)$$

is exact for all polynomials of degree at most 2.

$$\begin{cases} \varphi(x) = 1 \quad \Rightarrow \quad A + B + C = \int_0^h (1)dx = h & (1) \\ \\ \varphi(x) = x \quad \Rightarrow \quad A(-h) + C(h) = \int_0^h x dx = \frac{1}{2}h^2 & (2) \\ \\ \varphi(x) = x^2 \quad \Rightarrow \quad Ah^2 + Ch^2 = \int_0^h x^2 dx = \frac{1}{3}h^3 & (3) \end{cases}$$

Drop a factor of h from equation (2), and drop a factor of h from equation (3), to simplify to:

$$\begin{cases} \varphi(x) = 1 \quad \Rightarrow \quad A + B + C = \int_0^h (1)dx = h & (1) \\ \\ \varphi(x) = x \quad \Rightarrow \quad -A + C = \frac{1}{2}h & (2') \\ \\ \varphi(x) = x^2 \quad \Rightarrow \quad A + C = \frac{1}{3}h & (3') \end{cases}$$

By adding equations $(2')$ and $(3')$ together, we get:

$$2C = \frac{5}{6}h \quad \Rightarrow \quad \boxed{C = \frac{5}{12}h}$$

Putting this into $(3')$ gives us $\boxed{A = \frac{-1}{12}h}$

Then from (1) we get:

$$B = h - A - C \quad \Rightarrow \quad \boxed{B = \frac{8}{12}h}$$

So the approximating formula is:

$$\int_0^h \varphi(x)dx \approx -\frac{1}{12}h\,\varphi(-h)+\frac{8}{12}h\,\varphi(0)+\frac{5}{12}h\,\varphi(h)=\frac{h}{12}\Big[-\varphi(-h)+8\,\varphi(0)+5\,\varphi(h)\Big]$$

Now let $y' = f(x,y)$ be the differential equation under consideration. Then

$$y(x_{i+1})-y(x_i)=\int_{x_i}^{x_{i+1}} y'(x)dx$$

$$=\int_0^h y'(t+x_i)\,dt \qquad \text{change of variable} \begin{cases} t = x - x_i \\ dt = dx \end{cases}$$

$$\approx \frac{h}{12}\Big[-y'(-h+x_i)+8\,y'(x_i)+5\,y'(h+x_i)\Big]$$

$$=\frac{h}{12}\Big[-y'(x_{i-1})+8\,y'(x_i)+5\,y'(x_{i+1})\Big]$$

$$=\frac{h}{12}\Big[-f(x_{i-1},y(x_{i-1}))+8\,f(x_i,y(x_i))+5\,f(x_{i+1},y(x_{i+1}))\Big]$$

substitute approximate values of y for real
values of y as the real values are not accessible

$$\approx \frac{h}{12}\Big[-f(x_{i-1},y_{i-1})+8\,f(x_i,y_i)+5\,f(x_{i+1},y_{i+1})\Big]$$

In this way we get the **Third-Order Adams-Moulton Method**

$$y_{i+1}=y_i+\tfrac{h}{12}\Big[5\,f(x_{i+1},y_{i+1})+8\,f(x_i,y_i)-f(x_{i-1},y_{i-1})\Big]$$

4 Predictor-Corrector Methods (section 10.7)

The Adams-Moulton methods are implicit methods and therefore can not be solved as we do in the case of Adams-Bashforth methods. For example, for $y' = \sin(x+y)$, the third-order Adams-Moulton method becomes:

$$y_{i+1} = y_i + \tfrac{h}{12}\Big[5\,\sin(x_{i+1}+y_{i+1}) + 8\,\sin(x_i+y_i) - \sin(x_{i-1}+y_{i-1})\Big]$$

and this cannot be solved for y_{i+1}. So, to get the approximate value for y_{i+1} we use a two-step method called **predictor-corrector** method. In **step 1** we use an explicit method such as Adams-Bashforth method (or any other method) called the **predictor** to get an initial approximate value $y_{i+1}^{(1)}$:

$$y_{i+1}^{(1)} = y_i + \frac{h}{12}\Big[23\,f(x_i\,,\,y_i) - 16\,f(x_{i-1}\,,\,y_{i-1}) + 5\,f(x_{i-2}\,,\,y_{i-2})\Big]$$

and the in **step 2** which is called the **corrector** we improve the initial estimate; this is the implicit method under consideration:

$$y_{i+1}^{(k)} = y_i + \frac{h}{12}\Big[5\,f(x_{i+1}\,,\,y_{i+1}^{(k-1)}) + 8\,f(x_i\,,\,y_i) - f(x_{i-1}\,,\,y_{i-1})\Big]$$

and we continue until we get $|y_{i+1}^{(k)} - y_{i+1}^{(k-1)}| \le \varepsilon |y_{i+1}^{(k-1)}|$

So far 194 pages of typed materials, consisting of lecture notes, solutions to the lab questions, and solutions to the homework questions, have been given to the students