Numerical Differentiation

1 Finite Difference Formulas for the first derivative (Using Taylor Expansion technique) (section 8.3.1)

Suppose that $f(h) = g(h)h$ is a function of the variable h, and that as $h \to 0$ the function $g(h)$ remains bounded. This means that there exists some M such that $|g(h)| \leq M$ when $h \to 0$. Equivalently, we have $\left|\frac{f(h)}{h}\right| \leq M$ as $h \to 0$, equivalently $\frac{|f(h)|}{|h|} \leq M$ as $h \to 0$. By taking $|h|$ to the other side, we have $|f(h)| \leq M\,|h|$ as $h \to 0$. By this means that $f(h) = O(h)$. Similarly, if $f(h) = g(h)h^2$, where g remains bounded as $h \to 0$, then we have $f(h) = O(h^2)$, and so on.

Let f be any function defined on a vicinity of the fixed x , and let $h > 0$ representing a positive number close to zero (we are going to let $h \to 0$ later on). Using the Taylor expansion of the function f about the base point x, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

Move $f(x)$ to the left-hand side:

$$f(x+h) - f(x) = hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

Divide by h :

$$\begin{aligned} \tfrac{f(x+h)-f(x)}{h} &= f'(x) + \tfrac{h}{2!}f''(h) + \tfrac{h^2}{3!}f'''(x) + \cdots \\ &= f'(x) + h\left\{\tfrac{1}{2!}f''(x) + \tfrac{h^2}{3!}f'''(x) + \cdots\right\} \end{aligned}$$

The expression inside the braces tends to $\frac{1}{2}f''(x)$ as $h \to 0$, therefore remains bounded as $h \to 0$. So we can simply write the above equality as:

$$\frac{f(x+h) - f(x)}{h} = f'(x) + O(h)$$

By moving $O(h)$ to the other side we get

$$f'(x) = \frac{f(x+h) - f(x)}{h} - O(h)$$

But multiplying an $O(h)$ by some constant does not affect the status of $O(h)$, therefore $-O(h)$ is also is an $O(h)$, therefore this last equality can be written as

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h) \qquad (1)$$

in other words:

$$\mathbf{f'(x) \approx \frac{f(x+h)-f(x)}{h}} \qquad \textbf{with truncation error of order } \mathbf{O(h)} \qquad (2)$$

The expression $\frac{f(x+h)-f(x)}{h}$ is called a **two-point forward difference formula** for the first derivative. The difference $f'(x) - \frac{f(x+h)-f(x)}{h}$ which the truncation error is of order $O(h)$. So in the forward difference formula the truncation error is of order $O(h)$. We will see later on how to find approximations for $f'(x)$ in which the truncation error is of higher order of accuracy.

Example. The following data set of a values of a function f is given:

x	3	4	5
f(x)	27	64	125

What is the forward difference approximation for $f'(4)$?

Solution.

$$f'(4) \approx \frac{f(5) - f(4)}{5 - 4} = \frac{125 - 64}{1} = 61 \qquad \checkmark$$

Now in the equation (1) let us change h to $-h$ to get:

$$f'(x) = \frac{f(x-h) - f(x)}{-h} + O(-h)$$

But we have $O(-h) = O(h)$, therefore we may rewrite this above equality as:

$$f'(x) = \frac{f(x-h) - f(x)}{-h} + O(h)$$

equivalently

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h) \qquad (3)$$

The approximation

$$\mathbf{f'(x) \approx \frac{f(x)-f(x-h)}{h}}$$

is called the **two-point backward difference formula** for the first derivative

Example. The following data set of a values of a function f is given:

x	3	4	5
f(x)	27	64	125

What is the backward difference approximation for $f'(4)$?

Solution.

$$f'(4) \approx \frac{f(4) - f(3)}{4-3} = \frac{64-27}{1} = 37 \qquad \checkmark$$

Here we want to find a "central" difference formula . Let us start with the expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \qquad (4)$$

In this equality , replace h to $-h$

$$f(x-h) = f(x) - hf'(x) + \frac{(-h)^2}{2!}f''(x) + \frac{(-h)^3}{3!}f'''(x) + \cdots$$

Equivalently:

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \cdots \qquad (5)$$

By subtracting (5) from (4) we get:

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2h^3}{6}f'''(x) + \cdots$$

Upon dividing by $2h$:

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{2h^2}{6}f'''(x) + \cdots$$

So:

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2)$$

Equivalently:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

So:

$$\mathbf{f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}} \qquad \text{truncation error being of order of } \mathbf{O(h^2)}$$

This is called the **two-point central difference formula** for the first derivative. Note that the denominator $2h$ is just the difference between the endpoints $x+h$ and $x-h$. This approximating formula is better than the forward difference and the backward difference because the truncation error is smaller for small h's.

Example. The following data set of a values of a function f is given:

x	3	4	5
f(x)	27	64	125

What is the central difference approximation for $f'(4)$?

Solution. Here each step is 1 unit, therefore $2h = 2$. Hence:

$$f'(4) \approx \frac{f(5) - f(3)}{2} = \frac{125 - 27}{2} = 49 \qquad \checkmark$$

We can create forward difference formula of order $O(h^2)$. For this, let us start with the expansions:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \tag{6}$$

In this equality replace h by $2h$:

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \cdots$$

Equivalently

$$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x) + \cdots \tag{7}$$

By multiplying the identity (6) by 4 it becomes:

$$4f(x+h) = 4f(x) + 4hf'(x) + \frac{4h^2}{2!}f''(x) + \frac{4h^3}{3!}f'''(x) + \cdots \tag{6'}$$

Subtracting the identity (6') from the identity (7) gets rid of the term h^2:

$$f(x+2h) - 4f(x+h) = -3f(x) - 2hf'(x) + \frac{4h^3}{3!}f'''(x) + \cdots$$

which can be written as :

$$f(x+2h) - 4f(x+h) = -3f(x) - 2hf'(x) + O(h^3)$$

By moving $-3f(x)$ to the left-hand side we get:

$$f(x+2h) - 4f(x+h) + 3f(x) = -2hf'(x) + O(h^3)$$

Multiply both sides by -1 and reorder terms to get:

$-3f(x) + 4f(x+h) - f(x+2h) = 2hf'(x) + O(h^3)$

Dividing both sides by $2h$ gives us:

$$\frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} = f'(x) + O(h^2)$$

By taking $O(h^2)$ to the other side and ignoring a minus sign (which does not affect the O-notation):

$$f'(x) = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} + O(h^2)$$

$$\mathbf{f'(x) \approx \frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}} \qquad \text{truncation error being } \mathbf{O(h^2)}$$

This is called a **three-point forward difference formula** for the first derivative.

Example. The following data set of a values of a function f is given:

x	1.7	2.2	2.7
f(x)	27	64	125

What is the three-point forward difference approximation for $f'(1.7)$?

Solution. Here each step is 0.5 units, therefore $2h = 1$. Hence:

$$f'(3) = \frac{-3f(1.7) + 4f(2.2) - f(2.7)}{2h} = \frac{-3(27) + 4(64) - (125)}{1} = 50$$

Example (exercise 8.8 of the textbook). A particular finite difference formula for the first derivative of a function is

$$f'(x_i) \approx \frac{-f(x_{i+3}) + 9f(x_{i+1}) - 8f(x_i)}{6h}$$

where the points x_i , x_{i+1} , x_{i+2} , x_{i+3} are all equally spaced with step size h. What is the order of truncation or discretization error.

<u>Note</u>. I am going to give you a question like this in your next homework and in your midterm test, so learn this carefully.

<u>Solution</u>. Since the points are equally spaced , we set

$$\begin{cases} x_i = x \\ x_{i+1} = x + h \\ x_{i+3} = x + 3h \end{cases}$$

With these conventions, we then want to find the order of truncation error for estimating $f'(x)$ by

$$\frac{-f(x+3h) + 9f(x+h) - 8f(x)}{6h}$$

For this , we use the unterminated form of the Taylor expansion:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

Simplify to get:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \cdots \tag{1}$$

Change h to $3h$ in (1) to get:

$$f(x+3h) = f(x) + 3hf'(x) + \frac{(3h)^2}{2}f''(x) + \frac{(3h)^3}{6}f'''(x) + \cdots$$

Simplify:

$$f(x+3h) = f(x) + 3hf'(x) + \frac{9h^2}{2}f''(x) + \frac{9h^3}{2}f'''(x) + \cdots \tag{2}$$

Multiply the equality (1) by 9 and the equality (2) by -1 :

$$\begin{cases} 9f(x+h) = 9f(x) + 9hf'(x) + \frac{9h^2}{2}f''(x) + \frac{9h^3}{6}f'''(x) + \cdots \\ -f(x+3h) = -f(x) - 3hf'(x) - \frac{9h^2}{2}f''(x) - \frac{9h^3}{2}f'''(x) + \cdots \end{cases}$$

Now add the two equalities to get:

$$-f(x+3h)+9f(x+h)=8f(x)+6hf'(x)-\frac{18}{6}h^3f'''(x)+\cdots$$

Move $8f(x)$ to the left-hand side:

$$-f(x+3h)+9f(x+h)-8f(x)=6hf'(x)-\frac{18}{6}h^3f'''(x)+\cdots$$

Now divide by $6h$

$$\frac{-f(x+3h)+9f(x+h)-8f(x)}{6h}=f'(x)-\frac{3}{6}h^2f'''(x)+\cdots$$

$$\frac{-f(x+3h)+9f(x+h)-8f(x)}{6h}=f'(x)+O(h^2)$$

So

$$\mathbf{f'(x) \approx \frac{-f(x+3h)+9f(x+h)-8f(x)}{6h}} \qquad \textbf{with truncation error of order } \mathbf{O(h^2)}$$

Here I showed on the screen and discussed the "First Derivative" table on pages 317 and 318

2 Finite Difference Formulas for the second derivative (Using Taylor Expansion technique) (section 8.3.2)

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \cdots$$

$$\begin{aligned} f(x-h) &= f(x) - hf'(x) + \tfrac{(-h)^2}{2!}f''(x) + \tfrac{(-h)^3}{3!}f'''(x) + \tfrac{(-h)^4}{4!}f^{(4)}(x) + \cdots \\ &= f(x) - hf'(x) + \tfrac{h^2}{2!}f''(x) - \tfrac{h^3}{3!}f'''(x) + \tfrac{h^4}{4!}f^{(4)}(x) + \cdots \end{aligned}$$

Add together:

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \tfrac{h^4}{12}f^{(4)}(x) + \cdots$$

Move $2f(x)$ to the left-hand side:

$$f(x+h) + f(x-h) - 2f(x) = h^2 f''(x) + \frac{h^4}{12}f^{(4)}(x) + \cdots$$

Divide by h^2

$$\begin{aligned} \tfrac{f(x+h)+f(x-h)-2f(x)}{h^2} &= f''(x) + \tfrac{h^2}{12}f^{(4)}(x) + \cdots \\ &= f''(x) + h^2\left\{\tfrac{1}{12}f^{(4)}(x) + \cdots\right\} \\ &= f''(x) + O(h^2) \end{aligned}$$

So:

$$\mathbf{f''(x) \approx \frac{f(x-h)-2f(x)+f(x+h)}{h^2}}$$ **with truncation error of order $O(h^2)$**

This is called a **three-point central difference formula** for the second derivative.

Here I showed on the screen and discussed the "Second Derivative" and the "Third Derivative" tables on page 317

3 Richardson's Extrapolation (section 8.8)

Consider a procedure to approximate a value A with some value $D(h)$ that depends on h. For example we had

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + \cdots$$

Change h to $-h$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + \cdots$$

Subtract:

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2h^3}{3!}f'''(x) + \frac{2h^5}{5!}f^{(5)}(x) + \cdots$$

Divide by $2h$:

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{h^2}{3!}f'''(x) + \frac{h^4}{5!}f^{(5)}(x) + \cdots$$

Move around :

$$\underset{\substack{\uparrow \\ \mathbf{A}}}{f'(x)} = \underset{\substack{\uparrow \\ \mathbf{D(h)}}}{\tfrac{f(x+h)-f(x-h)}{2h}} - \tfrac{h^2}{3!}f'''(x) - \tfrac{h^4}{5!}f^{(5)}(x) - \cdots$$

in short we can write:

$$A = D(h) + \gamma_1 h^2 + \gamma_2 h^4 + \cdots \qquad (1)$$

in which the coefficients γ_1 and γ_2 , ... do not depend on h and therefore do not change when we replace h.

Now let us change h to $\frac{h}{2}$:

$$A = D(\frac{h}{2}) + \gamma_1 \left(\frac{h}{2}\right)^2 + \gamma_2 \left(\frac{h}{2}\right)^4 + \cdots$$

equivalently :

$$A = D(\frac{h}{2}) + \gamma_1 \frac{h^2}{4} + \gamma_2 \frac{h^4}{16} + \cdots$$

multiply by 4 :

$$4A = 4D(\frac{h}{2}) + \gamma_1 h^2 + \gamma_2 \frac{h^4}{4} + \cdots \qquad (2)$$

so in this discussion we have had these two:

$$\begin{cases} A &= D(h) + \gamma_1 h^2 + \gamma_2 h^4 + \cdots & (1) \\ 4A &= 4D(\frac{h}{2}) + \gamma_1 h^2 + \gamma_2 \frac{h^4}{4} + \cdots & (2) \end{cases}$$

subtract (1) from (2) :

$$3A = \left\{4D(\frac{h}{2}) - D(h)\right\} - \frac{3}{4}\gamma_2 h^4 + \cdots$$

divide by 3 :

$$A = \frac{1}{3}\left\{4D(\frac{h}{2}) - D(h)\right\} - \frac{1}{4}\gamma_2 h^4 + \cdots$$

so now we have found an approximation for A whose error is of a better order of $O(h^4)$

$$\mathbf{A \ \approx \ \frac{1}{3}\left\{4D(\frac{h}{2}) - D(h)\right\}} \qquad \textbf{with truncation error of order } \mathbf{O(h^4)}$$

This technique of improving the approximation is called **Richardson extrapolation**. We will use it in numerical integration as well.

In the particular case of $D(h) = \frac{f(x+h)-f(x-h)}{2h}$ we get:

$$D(\frac{h}{2}) = \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{2(\frac{h}{2})} = \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h}$$

so then :

$$\frac{1}{3}\left\{4\,D(\frac{h}{2}) - D(h)\right\} = \frac{1}{3}\left\{\frac{4\left\{f(x+\frac{h}{2}) - f(x-\frac{h}{2})\right\}}{h} - \frac{f(x+h) - f(x-h)}{2h}\right\}$$

$$= \frac{8f(x+\frac{h}{2}) - 8f(x-\frac{h}{2}) - f(x+h) + f(x-h)}{6h}$$

so :

$$A \;\approx\; \frac{f(x-h) - 8f(x-\frac{h}{2}) + 8f(x+\frac{h}{2}) - f(x+h)}{6h} + O(h^4)$$

Example. The derivative of the function $f(x) = \dfrac{1}{x+1} + \ln(x^2+1)$ is

$$f'(x) = -\frac{1}{(1+x)^2} + \frac{2x}{x^2+1}$$

whose value at $x = 0$ is -1. We want to approximate this value using the following methods, one is the original method and the second one is the improved method.

Method 1 : $\dfrac{f(x+h) - f(x-h)}{2h}$ of order $O(h^2)$

Method 2 : $\dfrac{f(x-h) - 8f(x-\frac{h}{2}) + 8f(x+\frac{h}{2}) - f(x+h)}{6h}$ of order $O(h^4)$.

We start with $h = 0.5$ and then using a "while" loop you reduce h in each step by 0.025. We continue until h is larger than or equal to 0.01.

h	method 1	method 2	error of method 1	error of method 2
0.500	-1.3333333333	-0.9777777778	0.3333333333	0.0222222222
0.475	-1.2913640032	-0.9825828109	0.2913640032	0.0174171891
0.450	-1.2539184953	-0.9864599091	0.2539184953	0.0135400909
0.425	-1.2204424104	-0.9895748885	0.2204424104	0.0104251115
0.400	-1.1904761905	-0.9920634921	0.1904761905	0.0079365079
0.375	-1.1636363636	-0.9940375414	0.1636363636	0.0059624586
0.350	-1.1396011396	-0.9955896481	0.1396011396	0.0044103519
0.325	-1.1180992313	-0.9967968592	0.1180992313	0.0032031408
0.300	-1.0989010989	-0.9977235041	0.0989010989	0.0022764959
0.275	-1.0818120352	-0.9984234343	0.0818120352	0.0015765657
0.250	-1.0666666667	-0.9989417989	0.0666666667	0.0010582011
0.225	-1.0533245556	-0.9993164600	0.0533245556	0.0006835400
0.200	-1.0416666667	-0.9995791246	0.0416666667	0.0004208754
0.175	-1.0315925210	-0.9997562536	0.0315925210	0.0002437464
0.150	-1.0230179028	-0.9998697919	0.0230179028	0.0001302081
0.125	-1.0158730159	-0.9999377529	0.0158730159	0.0000622471
0.100	-1.0101010101	-0.9999746842	0.0101010101	0.0000253158
0.075	-1.0056568196	-0.9999920339	0.0056568196	0.0000079661
0.050	-1.0025062657	-0.9999984326	0.0025062657	0.0000015674
0.025	-1.0006253909	-0.9999999023	0.0006253909	0.0000000977

As we see, in the improved method we get smaller errors for small h's.

Now we discuss the general form of the Richardson's extrapolation method.

Generally if we have an approximation method of order $O(h^m)$

$$A = D(h) + \gamma_1 \, h^m + O\left(h^{m+1}\right) \qquad (3)$$

then by changing h to $\frac{h}{2}$:

$$A = D(\frac{h}{2}) + \gamma_1\,(\frac{h}{2})^m + O\left((\frac{h}{2})^{m+1}\right)$$

equivalently

$$A = D(\frac{h}{2}) + \gamma_1 \frac{h^m}{2^m} + O\left(\frac{1}{2^{m+1}} h^{m+1}\right)$$

equivalently

$$A = D(\frac{h}{2}) + \gamma_1 \frac{h^m}{2^m} + O\left(h^{m+1}\right) \qquad \text{the O notation is not affected by constants}$$

multiply by 2^m

$$2^m\,A = 2^m\,D(\frac{h}{2}) + \gamma_1\,h^m + 2^m\,O\left(h^{m+1}\right)$$

equivalently

$$2^m\,A = 2^m\,D(\frac{h}{2}) + \gamma_1\,h^m + O\left(h^{m+1}\right) \qquad \text{the O notation is not affected by constants} \qquad (4)$$

So we have these two:

$$\begin{cases} A &= D(h) + \gamma_1\,h^m + O\left(h^{m+1}\right) \qquad (3) \\ 2^m\,A &= 2^m\,D(\frac{h}{2}) + \gamma_1\,h^m + O\left(h^{m+1}\right) \qquad (4) \end{cases}$$

Subtracting (3) from (4) gives:

$$(2^m - 1)A = \left(2^m\,D(\frac{h}{2}) - D(h)\right) + O\left(h^{m+1}\right)$$

Dividing by $2^m - 1$ gives:

$$A = \frac{1}{2^m - 1}\left(2^m\,D(\frac{h}{2}) - D(h)\right) + O\left(h^{m+1}\right)$$

So the combination

$$\frac{1}{2^m - 1}\left(2^m\, D(\frac{h}{2}) - D(h)\right)$$

gives a better approximation than $D(h)$.

Example. The method

$$\frac{f(x) - f(x - h^4)}{h^4}$$

has truncation order $O(h^4)$. Use the Richardson extrapolation method to improve it. What is the improved method?.

Solution.

4 Differentiation Formulas Using Lagrange Polynomials (section 8.5)

There are two main questions that we want to address in this section.

Question 1. We want to know how the book has figured out how to take particular combination in getting the numerical differentiation formulas in the Table 8-1 on pages 317, 318 , and 319 .

Question 2. How can we find a formula when the nodes are not equally spaced.

Both of these questions can be answered by means of Lagrange method for numerical differentiation. Supposing we have four unequally-spaced nodes $\{2\,,\,4\,,\,5\,,\,7\}$ and we want to find a numerical approximation for the derivatives $f''(0)$ and $f'(1.2)$ based on these nodes (a forward difference approximation) and the values of f at these nodes, for instance

x	2	4	5	7
f(x)	5	-5	-40	10

$\mathbf{x_k}$	$\mathbf{f}[\,] = \mathbf{f}()$	$\mathbf{f}[\,,\,]$	$\mathbf{f}[\,,\,,\,]$	$\mathbf{f}[\,,\,,\,,\,]$
2	5			
		$\frac{-5-5}{4-2} = -5$		
4	-5		$\frac{-35-(-5)}{5-2} = -10$	
		$\frac{-40-(-5)}{5-4} = -35$		$\frac{20-(-10)}{7-2} = 6$
5	-40		$\frac{25-(-35)}{7-4} = 20$	
		$\frac{10-(-40)}{7-5} = 25$		
7	10			

and then

$$p(x) = 5 - 5(x-2) - 10(x-2)(x-4) + 6(x-2)(x-4)(x-5)$$

Use the function `expand()` in Matlab to collect terms

$p(x) = 6x^3 - 76x^2 + 283x - 305$

For this, you put these two line on the command line:

$$\begin{cases} >> \ \texttt{syms x;} \\ >> \ \texttt{expand}(5 - 5 * (\texttt{x} - 2) - 10 * (\texttt{x} - 2) * (\texttt{x} - 4) + 6 * (\texttt{x} - 2) * (\texttt{x} - 4) * (\texttt{x} - 5)) \end{cases}$$

Now differentiate this polynomial twice.

$p'(x) = 18x^2 - 152x + 283$

$p''(x) = 36x - 152$

Now we can use $p''(x)$ to approximate any value of $f''(x)$ and not just $f''(0)$. This is an advantage of the Lagrange method over the Taylor expansion.

$f''(0) \approx p''(0) = -152$

$f'(1.2) \approx p'(1.2) = 126.52$

Note 1. Advantage of Lagrange Method over Taylor Method:

1. It can be used for unequally-spaced nodes.
2. It gives approximation for the derivative (of any order) at any point (and not just the nodes).

Note 2.: If one applies the Lagrange method for equally spaced noted , the he gets the difference formulas we have learned before.

Note 3.: Based on the Lagrange Method, if some nodes $\{x_1 \,, \dots, \, x_n\}$ and some values $\{f(x_1) \,, \dots, \, f(x_n)\}$ of a function f are given, then to approximate $f'(x)$ at any point x (that could by other than the nodes), we fist find the interpolating polynomial $p(x)$ associated with those values and then

$$\begin{cases} f'(x) & \approx & p'(x) \\ f''(x) & \approx & p''(x) \\ & \vdots & \end{cases}$$

Now here, in the following example, is another way of doing the same problem but without actually finding the interpolating polynomial $p(x)$.

Example. Find a, b, and c such that the approximation

$$f'(0) \approx af(0) + bf(1) + cf(2)$$

is exact for polynomials of degree less than or equal to 2.

Solution. We force the above approximation to be exact for the polynomials 1 , x , x^2 that generate all polynomials of degree less than or equal to 2 through linear combinations:

$$af(0) + bf(1) + cf(2) = f'(0) \qquad \text{for} \quad f(x) = 1 \ , \ f(x) = x \ , \ f(x) = x^2$$

$$\begin{cases} f(x) = 1 & \Rightarrow & a + b + c = 0 & (1) \\ f(x) = x & \Rightarrow & 0 + b + 2c = 1 & (2) \\ f(x) = x^2 & \Rightarrow & 0 + b + 4c = 0 & (3) \end{cases}$$

♣ $\quad (2) \quad \Rightarrow \quad b = 1 - 2c \qquad (4)$

♣ $\quad (3)\,,\,(4) \quad \Rightarrow \quad (1 - 2c) + 4c = 0 \quad \Rightarrow \quad 1 + 2c = 0$

$$\Rightarrow \quad c = -\frac{1}{2} \quad \Rightarrow \quad b = 1 - 2c = 1 + 1 = 2$$

♣ $\quad (1) \quad \Rightarrow \quad a = -b - c = -2 + \frac{1}{2} = -\frac{3}{2}$

So:

$$\begin{cases} a = -\frac{3}{2} \\ b = 2 \\ c = -\frac{1}{2} \end{cases} \qquad \Rightarrow \qquad f'(0) \approx -\frac{3}{2}f(0) + 2f(1) - \frac{1}{2}f(2) \qquad \checkmark$$

Example. Now here we see how to get the formula "three-point forward difference"

$$g'(x) \approx \frac{-3g(x) + 4g(x+h) - g(x+2h)}{2h}$$

by referring to the approximation in the last example. In fact let $f(t) = g(x + th)$ and note that from the chain rule we have $f'(t) = h\, g'(x + th)$. Especially $f'(0) = h\, g'(x)$. Then

$$f'(0) \approx -\frac{3}{2}f(0) + 2f(1) - \frac{1}{2}f(2) \qquad \Rightarrow$$

$$h\, g'(x) = -\frac{3}{2}g(x) + 2g(x+h) - \frac{1}{2}g(x+2h) \qquad \stackrel{\text{divide by h}}{\Rightarrow}$$

$$g'(x) \approx \frac{-3g(x) + 4g(x+h) - g(x+2h)}{2h} \qquad \checkmark$$

Note. As we have seen,