Higher-Order Partial Derivatives
section 12.5

For the function $f(x, y) = x^3y^2 + ye^x$ we have

$$\frac{\partial f}{\partial x} = 3x^2y^2 + ye^x \qquad \frac{\partial f}{\partial y} = 2x^3y + e^x$$

Now we may calculate the second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2y^2 + ye^x\right) = 6xy^2 + ye^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2y^2 + ye^x\right) = 6x^2y + e^x$$

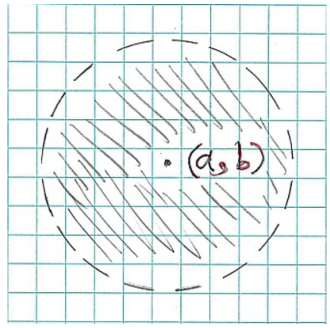
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(2x^3y + e^x\right) = 6x^2y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(2x^3y + e^x\right) = 2x^3$$

Note. As we can see in this example, we have $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. This is special case of the following theorem:

Theorem. If the functions f , $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$ exist on all points of a disk centered at (a, b) (i.e. for the points close to (a, b)), and if these functions are continuous at (a, b), then we must have:

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b)$$



Note. Note the way higher-order derivatives are shown in the notation ∂ and in the "subscript" notation. In subscript notation, f_{xy} is meant to be $(f_x)_y$. For example f_{yxxyx} is the same as $\frac{\partial^5 f}{\partial x \partial y \partial x^2 \partial y}$. For the notation $\frac{\partial^5 f}{\partial x \partial y \partial x^2 \partial y}$ the order they appear is from right to left, but in the notation f_{yxxyx} they appear from left to right.

Example. For the function $f(x,y) = x^3y^2 + ye^x$ of the previous example calculate f_{xxy}.

Solution.

$$f_{xxy} = \frac{\partial^3 f}{\partial y \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial y}\left(6xy^2 + ye^x\right) = 12xy + e^x$$

Definition. A function $f(x,y)$ is said to be **harmonic**on a region R if

- it has continuous second-order partial derivatives in R , and
- it satisfies the **Laplace's Equation** :

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

  at all points of the region R

Note. A similar definition holds for function $\mathrm{f(x,y,z)}$ but in this case the Laplace's equation is

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0$$

Example (section 12.5 exercise 27). Show that the function $\mathrm{f(x,y,z)}=\frac{1}{\sqrt{\mathrm{x^2+y^2+z^2}}}$ is harmonic in the region

$$R=\Big\{(x,y,z)\ :\ \ (x,y,z)\neq(0,0,0)\Big\} \qquad \text{the space minus the origin}$$

Solution.

$\mathrm{f(x,y,z)}=(\mathrm{x^2+y^2+z^2})^{-\frac{1}{2}}$

$\mathrm{f_x}=-\frac{1}{2}(\mathrm{x^2+y^2+z^2})^{-\frac{3}{2}}(2\mathrm{x})=-\mathrm{x}(\mathrm{x^2+y^2+z^2})^{-\frac{3}{2}}$

$$
\begin{aligned}
f_{xx} &= \frac{\partial}{\partial x}(-x)\left\{(x^2+y^2+z^2)^{-\frac{3}{2}}\right\}+(-x)\frac{\partial}{\partial x}\left\{(x^2+y^2+z^2)^{-\frac{3}{2}}\right\} \\
&= -\left\{(x^2+y^2+z^2)^{-\frac{3}{2}}\right\}+(-x)\left\{\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}(2x)\right\} \\
&= -\left\{(x^2+y^2+z^2)^{-\frac{3}{2}}\right\}+3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} \\
&= \frac{-1}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}+\frac{3x^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}} \\
&= \frac{-(x^2+y^2+z^2)+3x^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}} \\
&= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}}
\end{aligned}
$$

In a similar manner (use symmetry) one can show:

$$
f_{yy}=\frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}} \qquad f_{zz}=\frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}}
$$

Now adding up these values, one gets:

$$
f_{xx}+f_{yy}+f_{zz}=0
$$

Further note the second order derivatives

$$\begin{cases} f_{xx} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}} \\ \\ f_{yy} = \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}} \\ \\ f_{zz} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}} \end{cases}$$

are continuous in the region R, so we infer that this function is harmonic over the region R.