Section 12.7

A brief on transformations

consider the transformation (**another name for it**: change of variable)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

The determinant

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

is called the **Jacobian determinant** of (x,y) with respect to (r,θ) , and is denoted by $\frac{\partial(x,y)}{\partial(r,\theta)}$. Let us calculate this determinant:

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

Definition . If $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ is a transformation , then the determinant

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is called the **Jacobian determinant** of the functions x and y in terms of the variables u and v. This Jacobian is a function of the variables u and v see the next example).

Example . Find the Jacobian of the transformation $\begin{cases} x = u + 3v^2 \\ y = u^2 - uv \end{cases}$.

Solution .

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 6v \\ 2u-v & -u \end{vmatrix}$$

$$= (-u) - (6v)(2u-v)$$

$$= -u - 12uv + 6v^2 \quad \text{(this is a function of u and v)}$$

Note . Sometimes we are interested in a particular value of the Jacobian. See the next example:

Example . In the previous example, what is the value of the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ at the point where $(u,v) = (1,-1)$.

Solution .

$$\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u,v)=(1,-1)} = (-u - 12uv + 6v^2)\big|_{(u,v)=(1,-1)} = 17$$

Fact . It is true that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$$

This is a useful formula when it is difficult or impossible to find the inverse transformation

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

Example . For the transformation

$$\begin{cases} x = u^3 + uv + v \\ y = v^3 + uv + v \end{cases}$$

calculate $\frac{\partial(u,v)}{\partial(x,y)}$ at the point where $(u,v) = (1,-1)$.

Solution . We are not able to calculate (u,v) in terms of (u,v) (the inverse transformation) therefore we cannot calculate $\frac{\partial(\mathrm{u},\mathrm{v})}{\partial(\mathrm{x},\mathrm{y})}$ directly. So we use the formula

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$$

By substituting $\mathrm{u} = 1$ and $\mathrm{v} = -1$ we get $\mathrm{x} = -1$ and $\mathrm{y} = -3$.

$$\left.\frac{\partial(\mathrm{x},\mathrm{y})}{\partial(\mathrm{u},\mathrm{v})}\right| = \begin{vmatrix} \frac{\partial \mathrm{x}}{\partial \mathrm{u}} & \frac{\partial \mathrm{x}}{\partial \mathrm{v}} \\ \frac{\partial \mathrm{y}}{\partial \mathrm{u}} & \frac{\partial \mathrm{y}}{\partial \mathrm{v}} \end{vmatrix} = \begin{vmatrix} 3\mathrm{u}^2+\mathrm{v} & \mathrm{u}+1 \\ \mathrm{v} & 3\mathrm{v}^2+\mathrm{u}+1 \end{vmatrix}$$

$$\left.\frac{\partial(\mathrm{x},\mathrm{y})}{\partial(\mathrm{u},\mathrm{v})}\right|_{(\mathrm{u},\mathrm{v})=(1,-1)} = \begin{vmatrix} 2 & 2 \\ -1 & 5 \end{vmatrix} = 12 \quad \Rightarrow \quad \frac{\partial(\mathrm{u},\mathrm{v})}{\partial(\mathrm{x},\mathrm{y})} = \frac{1}{\frac{\partial(\mathrm{x},\mathrm{y})}{\partial(\mathrm{u},\mathrm{v})}} = \frac{1}{12}$$

Definition . For a 3×3 transformation

$$\begin{cases} \mathrm{x} = \mathrm{x}(\mathrm{u},\mathrm{v},\mathrm{w}) \\ \mathrm{y} = \mathrm{y}(\mathrm{u},\mathrm{v},\mathrm{w}) \\ \mathrm{z} = \mathrm{z}(\mathrm{u},\mathrm{v},\mathrm{w}) \end{cases}$$

the Jacobian (determinant) is defined by

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial \mathrm{x}}{\partial \mathrm{u}} & \frac{\partial \mathrm{x}}{\partial \mathrm{v}} & \frac{\partial \mathrm{x}}{\partial \mathrm{w}} \\ \frac{\partial \mathrm{y}}{\partial \mathrm{u}} & \frac{\partial \mathrm{y}}{\partial \mathrm{v}} & \frac{\partial \mathrm{y}}{\partial \mathrm{w}} \\ \frac{\partial \mathrm{z}}{\partial \mathrm{u}} & \frac{\partial \mathrm{z}}{\partial \mathrm{v}} & \frac{\partial \mathrm{z}}{\partial \mathrm{w}} \end{vmatrix}$$

This quantity enjoys a similar identity as in the two dimensional case:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}}$$

Note . In general if we have an $\mathrm{n} \times \mathrm{n}$ system of equations such as:

$$\begin{cases} \mathrm{y}_1 & = & \mathrm{f}_1(\mathrm{x}_1,,\ldots,\mathrm{x}_\mathrm{n}) \\ \mathrm{y}_2 & = & \mathrm{f}_2(\mathrm{x}_1,,\ldots,\mathrm{x}_\mathrm{n}) \\ & \vdots & \\ \mathrm{y}_\mathrm{n} & = & \mathrm{f}_\mathrm{n}(\mathrm{x}_1,,\ldots,\mathrm{x}_\mathrm{n}) \end{cases}$$

then its Jacobian (determinant) is defined by:

$$\frac{\partial(y_1, , \ldots, y_n)}{\partial(x_1, , \ldots, x_n)} = \begin{vmatrix} \frac{\partial \mathrm{y_1}}{\partial \mathrm{x_1}} & \frac{\partial \mathrm{y_1}}{\partial \mathrm{x_2}} & \cdots & \frac{\partial \mathrm{y_1}}{\partial \mathrm{x_n}} \\ \frac{\partial \mathrm{y_2}}{\partial \mathrm{x_1}} & \frac{\partial \mathrm{y_2}}{\partial \mathrm{x_2}} & \cdots & \frac{\partial \mathrm{y_2}}{\partial \mathrm{x_n}} \\ & & & \\ \vdots & \vdots & & \vdots \\ \frac{\partial \mathrm{y_n}}{\partial \mathrm{x_n}} & \frac{\partial \mathrm{y_n}}{\partial \mathrm{x_2}} & \cdots & \frac{\partial \mathrm{y_n}}{\partial \mathrm{x_n}} \end{vmatrix}$$

the first row being the derivatives of y_1

Implicit Differentiation

Suppose we have a system of m equations with $m+n$ unknowns:

$$\begin{cases} F_1(x_1,\ldots,x_n,y_1,\ldots,y_m) &= 0 \\ F_2(x_1,\ldots,x_n,y_1,\ldots,y_m) &= 0 \\ &\vdots \\ F_m(x_1,\ldots,x_n,y_1,\ldots,y_m) &= 0 \end{cases}$$

We consider the n number of variables (the difference between $m+n$ and m, i.e. the difference between the number of unknowns and the number of equations) as <u>independent</u> variables. For simplicity suppose that $x_1,\ldots,x_n$ are the independent variables, and so $y_1,\ldots,y_m$ are the dependent variables. Then the first order partial derivatives of the dependent variables with respect to the independent variables are calculated through:

$$\frac{\partial y_i}{\partial x_k} = -\frac{\frac{\partial(F_1,\ldots,F_m)}{\partial(y_1,\ldots,x_k,\ldots,y_m)}}{\frac{\partial(F_1,\ldots,F_m)}{\partial(y_1,\ldots,y_i,\ldots,y_m)}} \qquad x_k \text{ is sat the i-th place in th numerator}$$

The denominator is the Jacobian with respect to the dependent variables.

<u>**Example**</u> . Consider the system

$$\begin{cases} xy^2+xzu+yv^2 &= 3 \\ x^3yz+2xv-u^2v^2 &= 2 \end{cases}$$

We must consider as many as $5-2=3$ variables as independent ones. Let us consider the variables (u,v) as the dependent variables and the variables (x,y,z) as the independent ones. Find $\frac{\partial v}{\partial y}$ at the point with coordinates $(x,y,z,u,v)=(1,1,1,1,1)$.

<u>**Solution**</u> . Set (the first step is the naming of the equations):

$$\begin{cases} F(x,y,z,u,v) &= xy^2 + xzu + yv^2 - 3 \\ G(x,y,z,u,v) &= x^3yz + 2xv - u^2v^2 - 2 \end{cases}$$

Then, at the point with coordinates $(1,1,1,1,1)$ we have:

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} xz & 2yv \\ -2uv^2 & 2x - 2u^2v \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 4$$

$$\frac{\partial(F,G)}{\partial(u,y)} = \begin{vmatrix} xz & 2xy + v^2 \\ -2uv^2 & x^3z \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 7$$

$$\frac{\partial v}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}} = -\frac{7}{4}$$

Example. Consider 4 variables which are tangled together through:

$$\begin{cases} u &= x^2 + xy - y^2 \\ v &= 2xy + y^2 \end{cases}$$

(i) Consider x and y as functions of (u,v). Find $\frac{\partial x}{\partial u}$ at the point with $(x,y) = (2,-1)$.

(ii) Now consider x and v as functions of y and u. Find $\frac{\partial x}{\partial u}$ at the point with $(x,y) = (2,-1)$.

Solution to part (i). We write the equations in the form:

$$\begin{cases} F(x,y,u,v) &= x^2 + xy - y^2 - u \\ G(x,y,z,u,v) &= 2xy + y^2 - v \end{cases}$$

At the point with characteristics $x = 2$ and $y = -1$ we have:

$$\frac{\partial x}{\partial u} = -\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(x,y)}} = \frac{\begin{vmatrix} -1 & x-2y \\ 0 & 2x+2y \end{vmatrix}}{\begin{vmatrix} 2x+y & x-2y \\ 2y & 2x+2y \end{vmatrix}} = -\frac{\begin{vmatrix} -1 & 4 \\ 0 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & 2 \end{vmatrix}} = -\frac{-2}{14} = \frac{1}{7}$$

Solution to part (ii) . Now we are considering (x, v) as dependent variables. So:

$$\frac{\partial x}{\partial u} = -\frac{\frac{\partial(F,G)}{\partial(u,v)}}{\frac{\partial(F,G)}{\partial(x,v)}} = \frac{\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 2x+y & 0 \\ 2y & -1 \end{vmatrix}} = -\frac{\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 0 \\ -2 & -1 \end{vmatrix}} = -\frac{1}{-3} = \frac{1}{3}$$

Note . Consider an equation $F(x, y) = 0$ which actually describes a curve in the plane defined implicitly (such as $x^3y^2 - 2xy + 5 = 0$). If y is considered as a function of x, then by applying the implicit differentiation formula we learned for the general case, we will have:

$$\frac{\partial y}{\partial x} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial y}\right)}$$

Note . Consider an equation $F(x, y, z) = 0$. If z is considered as a function of (x, y), then by applying the implicit differentiation formula we learned for the general case, we will have:

$$\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)} \qquad \frac{\partial z}{\partial y} = -\frac{\left(\frac{\partial F}{\partial y}\right)}{\left(\frac{\partial F}{\partial z}\right)}$$

Example (section 12.7 exercise 1) . Consider y as a function of x in the equation $x^3y^2 - 2xy + 5 = 0$. Find $\frac{dy}{dx}$.

Solution.

First Method. Put $F(x,y) = x^3y^2 - 2xy + 5$

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial y}\right)} = -\frac{3x^2y^2 - 2y}{2x^3y - 2x}$$

Second Method. Differentiate both sides of $x^3y^2 - 2xy + 5 = 0$ with respect to x considering y as a function of x (this is an elementary Calculus subject):

$$\Big\{\{x^3\}'\{y^2\} + \{x^3\}\{y^2\}'\Big\} - 2\Big\{\{x\}'\{y\} + \{x\}\{y\}'\Big\} = 0$$

$$\Big\{\{3x^2\}\{y^2\} + \{x^3\}\{2yy'\}\Big\} - 2\Big\{\{1\}\{y\} + \{x\}\{y'\}\Big\} = 0$$

$$(3x^2y^2 - 2y) + (2x^3y - 2x)y' = 0$$

$$y' = -\frac{3x^2y^2 - 2y}{2x^3y - 2x}$$

Note. As you see, the first method is much simpler.

Example (section 12.7 exercise 5). The variable z is defined implicitly as a function of x and y through $x^2\sin z - y\,e^z = 2x$. Find $\frac{\partial z}{\partial x}$.

Solution. **First Method**. Put $F = x^2\sin z - y\,e^z - 2x = 0$

$$\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)} = -\frac{2x\sin z - 2}{x^2\cos z - y\,e^z} = \frac{2 - 2x\sin z}{x^2\cos z - y\,e^z}$$

Second Method . Differentiate the equation $x^2 \sin z - y e^z = 2x$ with respect to x keeping in mind that the derivative of y with respect to x is zero because in this question it is assumed that x and y are independent and z is a function of them.

$$\left\{\{x^2\}'\{\sin z\} + \{x^2\}\{\sin z\}'\right\} - \left\{\{y\}'\{e^z\} + \{y\}\{e^z\}'\right\} = \{2x\}'$$

$$\left\{\{2x\}\{\sin z\} + \{x^2\}\{(\cos z) z'\}\right\} - \left\{\{0\}\{e^z\} + \{y\}\{e^z z'\}\right\} = 2$$

$$\left\{\{2x\}\{\sin z\} + \{x^2\}\{(\cos z) z'\}\right\} + \{y\}\{e^z z'\}\Big\} = 2$$

$$2x \sin z + \left(x^2 \cos z + y e^z\right) z' = 2$$

$$z' = \frac{2 - 2x \sin z}{x^2 \cos z + y e^z}$$

Example (section 12.7 exercise 17) . Find $\left(\frac{\partial s}{\partial u}\right)_v$ if $s = x^2 + y^2$, and x and y are functions of u and v defined by

$$u = x^2 - y^2 \qquad v = x^2 - y$$

Solution . We are assuming that

$$\begin{cases} s = s(x,y) \\ x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$\left(\frac{\partial s}{\partial u}\right)_v = \frac{\partial s}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial s}{\partial y}\frac{\partial y}{\partial u} = (2x)\frac{\partial x}{\partial u} + (2y)\frac{\partial y}{\partial u} \qquad (*)$$

Now to calculate $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ we need to apply the implicit differentiation technique because x and y are not given explicitly in terms of u and v:

$$\begin{cases} F(x,y,u,v) = x^2 - y^2 - u \\ G(x,y,u,v) = x^2 - y - v \end{cases}$$

$$\frac{\partial x}{\partial u} = -\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{vmatrix}} = -\frac{\begin{vmatrix} -1 & -2y \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 2x & -2y \\ 2x & -1 \end{vmatrix}} = -\frac{1}{-2x+4xy} = \frac{1}{2x-4xy}$$

$$\frac{\partial y}{\partial u} = -\frac{\frac{\partial(F,G)}{\partial(x,u)}}{\frac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial u} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial u} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{vmatrix}} = -\frac{\begin{vmatrix} 2x & -1 \\ 2x & 0 \end{vmatrix}}{\begin{vmatrix} 2x & -2y \\ 2x & -1 \end{vmatrix}} = -\frac{2x}{-2x+4xy} = \frac{2x}{2x-4xy}$$

Now by putting these into (*) , one gets:

$$\left(\frac{\partial s}{\partial u}\right)_v = (2x)\left(\frac{1}{2x-4xy}\right) + (2y)\left(\frac{2x}{2x-4xy}\right) = \frac{2x+2y}{2x-4xy} = \frac{x+y}{x-2xy} \qquad \checkmark$$