Limits and Continuity:
section 12.2

Definition. Let $f(x,y)$ be a function with domain D, and let (a,b) be a point in the plane. We write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for each $\varepsilon > 0$ there exists some $\delta > 0$ such that if $(x,y) \in D$ and $0 < \|(x,y)-(a,b)\| < \delta$, then $|f(x,y)-L| < \varepsilon$.

Note. To show $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ we might equivalently show that $\lim\limits_{(x,y)\to(a,b)} |f(x,y)-L| = 0$

Algebraic Properties of the Limit Operator.

(i) $\lim\limits_{(x,y)\to(a,b)} \Big[c\,f(x,y)\Big] = c \lim\limits_{(x,y)\to(a,b)} f(x,y)$ c being a constant

(ii) $\lim\limits_{(x,y)\to(a,b)} \Big[f(x,y) \pm g(x,y)\Big] = \lim\limits_{(x,y)\to(a,b)} f(x,y) \pm \lim\limits_{(x,y)\to(a,b)} g(x,y)$

(iii) $\lim\limits_{(x,y)\to(a,b)} \Big[f(x,y)g(x,y)\Big] = \lim\limits_{(x,y)\to(a,b)} f(x,y) \lim\limits_{(x,y)\to(a,b)} g(x,y)$

(iv) $\lim\limits_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim f(x,y)}{\lim g(x,y)}$

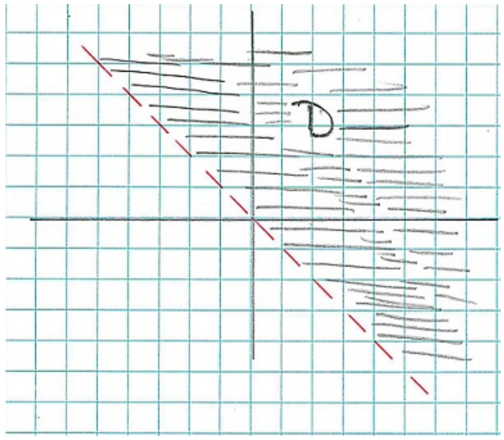
Definition. If (a,b) is in the domain of f, then we say that f is continuous at (a,b) if the following holds:

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Note. Sum, difference, and product of continuous functions is continuous. The quotient $\frac{f(x,y)}{g(x,y)}$ of two continuous functions at (a,b) is continuous at (a,b) provided that $g(a,b) \neq 0$. Composite of continuous functions is continuous.

Example. The function $f(x,y) = \frac{\sin(x^2-y)}{\sqrt{x+y}}$ has the domain consisting of the points (x,y) satisfying $x+y>0$:

$$D_f = \Big\{(x,y) \mid x+y>0\Big\}$$



This function is continuous everywhere on its domain because it is the quotient of two continuous functions wit the denominator being nonzero on D. Therefore, one can write

$$\lim_{\substack{x\to 3\\ y\to 1}} \frac{\sin(x^2-y)}{\sqrt{x+y}} \underset{\substack{\uparrow\\ \textbf{continuity}}}{=} \frac{\sin(9-1)}{\sqrt{3+1}} = \frac{\sin(8)}{2}$$

Note. The functions $f(x,y) = x^2y - y$ and $g(x,y) = x + y$ are examples of polynomials. The polynomial f is of degree 3 , and g is of degree 1 (the largest degree present). In general, a polynomial is a combination of expressions $x^m y^n$ where $m \geq 0$ and $n \geq 0$ are non-negative integers. Polynomials have the plane as their domain. Polynomials are continuous everywhere.

Example. Find the limit $\lim_{(x,y)\to(1,0)} \sin\left(\frac{1+x^2}{x^2+xy+1}\right)$, or show that it does not exist.

Solution.

$$\begin{aligned}
\lim_{(x,y)\to(1,0)} \sin\left(\frac{1+x^2}{x^2+xy+1}\right) &= \sin\left(\lim_{(x,y)\to(1,0)} \frac{1+x^2}{x^2+xy+1}\right) \\
&= \sin\left(\frac{1+1}{1+0+1}\right) \\
&= \sin(1)
\end{aligned}$$

the sine function is continuous

so the limit operator can go inside the sine function

Example. Find the limit $\lim\limits_{(x,y)\to(3,3)} \frac{\sqrt{x+1}-\sqrt{y+1}}{x-y}$, or show that it does not exist.

Solution. This is of the form $\frac{0}{0}$. We rationalize it to find the answer:

$$\begin{aligned}
\lim_{(x,y)\to(3,3)} \frac{\sqrt{x+1}-\sqrt{y+1}}{x-y} &= \lim_{(x,y)\to(3,3)} \frac{\sqrt{x+1}-\sqrt{y+1}}{x-y}\,\frac{\sqrt{x+1}+\sqrt{y+1}}{\sqrt{x+1}+\sqrt{y+1}} \\
&= \lim_{(x,y)\to(3,3)} \frac{(x+1)-(y+1)}{(x-y)(\sqrt{x+1}+\sqrt{y+1})} \\
&= \lim_{(x,y)\to(3,3)} \frac{x-y}{(x-y)(\sqrt{x+1}+\sqrt{y+1})} \\
&= \lim_{(x,y)\to(3,3)} \frac{1}{\sqrt{x+1}+\sqrt{y+1}} \qquad \text{drop} \quad x-y \\
&= \frac{1}{\sqrt{3+1}+\sqrt{3+1}} = \frac{1}{4}
\end{aligned}$$

Example (section 12.2 exercise 35). Calculate the limit $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ if it exists.

Solution. This is of the form $\frac{0}{0}$. By putting $u = x^2+y^2$, we have $\lim\limits_{(x,y)\to(0,0)} u = 0$. so then

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{u\to 0} \frac{\sin u}{u} = 1 \qquad \text{this limit is proved in elementary Calculus}$$

Example (section 12.2 exercise 26). Find all points of discontinuity of the function $f(x,y) = \frac{x+y}{x^2y+xy^2}$

Solution. We have $f(x,y) = \dfrac{x+y}{xy(x+y)}$.

Note that the points (x,y) which satisfy either of $xy = 0$ or $x+y = 0$ are not in the domain of f so we cannot consider them as some points of discontinuity. So, the function is indeed continuous at any point of its domain. Note that the solution given in the Solution Manual is wrong.

Example (section 12.2 exercise 27). Evaluate the limit $\lim_{(x,y)\to(a,a)} \left[\cos(x+y) - \sqrt{1-\sin^2(x+y)}\right]$ where $0 \le a \le \frac{\pi}{2}$.

Solution.

$$\begin{aligned}
\lim_{(x,y)\to(a,a)} \left[\cos(x+y) - \sqrt{1-\sin^2(x+y)}\right] &= \cos(a+a) - \sqrt{1-\sin^2(a+a)} \qquad \text{continuity of the functions sine and cosine} \\
&= \cos(2a) - \sqrt{1-\sin^2(2a)} \\
&= \cos(2a) - \sqrt{\cos^2(2a)} \\
&= \cos(2a) - |\cos(2a)| \\
&= \begin{cases} \cos(2a) - \cos(2a) = 0 & \text{if} \quad 0 \le 2a \le \frac{\pi}{2} \\ \cos(2a) + \cos(2a) = 2\cos(2a) & \text{if} \quad \frac{\pi}{2} \le 2a \le \pi \end{cases}
\end{aligned}$$

Example (section 12.2 exercise 11). Calculate the limit $\lim_{(x,y)\to(2,1)} \frac{x^2-y^2}{x-y}$ if it exists.

Solution. This is not of the form $\frac{0}{0}$.

$$\lim_{(x,y)\to(2,1)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(2,1)} \frac{(x-y)(x+y)}{x-y} = \lim_{(x,y)\to(2,1)} (x+y) = 3$$

Example (section 12.2 exercise 13). Evaluate the limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y}$ if it exists.

Solution. This limit is of the indeterminate form $\frac{0}{0}$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(0,0)} \frac{(x-y)(x+y)}{x-y} = \lim_{(x,y)\to(0,0)} (x+y) \underset{\substack{\uparrow \\ \text{continuity of polynomials}}}{=} 0$$

Note. we learned in elementary Calculus that if a limit exists then the left-hand limit and right-hand limit exist and are equal. A similar result holds for multivariable functions: If a limit $\lim\limits_{(x,y)\to(a,b)} f(x,y)$, then the limit on any path ending at (a,b) must exist and these limits all must be equal.

Example (section 12.2 exercise 29). Evaluate the limit $\lim\limits_{(x,y)\to(0,0)} \frac{x^6-2y^2}{3x^6+y^2}$ if it exists.

Solution. We take the following two paths:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{on the path } x=0}} \frac{x^6-2y^2}{3x^6+y^2} = \lim_{x\to 0} \frac{-2y^2}{y^2} = \frac{-1}{2}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{on the path } y=0}} \frac{x^6-2y^2}{3x^6+y^2} = \lim_{x\to 0} \frac{x^6}{3x^6} = \frac{1}{3}$$

Since we get different values on different paths, the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^6-2y^2}{3x^6+y^2}$ does not exist.

Note . The next three examples use the concept of “equivalence”.

Example . Evaluate the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{2x^2+y^2}$ or show that it does not exist.

Solution . As we know form elementary Calculus, we have $\lim_{y\to 0} \frac{\sin y}{y} = 1$, therefore the expressions $\sin y$ and y are equivalent as $y \to 0$, in the sense that as long as multiplication or division is concerned, we can use y for $\sin y$. So now, instead of calculating the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{2x^2+y^2}$ we may calculate this limit: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{2x^2+y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{x^2+y^2}{2x^2+y^2} = \lim_{x\to 0} \frac{x^2+x^2}{2x^2+x^2} = \lim_{x\to 0} \frac{2x^2}{3x^2} = \frac{2}{3}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2+y^2}{2x^2+y^2} = \lim_{x\to 0} \frac{x^2+0}{2x^2+0} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Conclusion: the limit does not exists as we get different values on different paths.

Example . Evaluate the limit $\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{3x^2+y^2}$ or show that it does not exist.

Solution . We know that $\lim_{y\to 0}\cos y = 1$, therefore we may substitute 1 for $\cos y$ in evaluating the limit ($\cos y$ and 1 are equivalent), and we may study the following limit instead:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2+y^2}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{xy}{3x^2+y^2} = \lim_{x\to 0} \frac{x^2}{3x^2+x^2} = \lim_{x\to 0} \frac{x^2}{4x^2} = \frac{1}{4}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{xy}{3x^2+y^2} = \lim_{x\to 0} \frac{0}{3x^2+0} = 0$$

Conclusion: the limit does not exists as we get different values on different paths.

Here are two examples where changing to polar coordinates will help:

Example . Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2} = 0$

Solution .

Method 1 (by changing to polar coordinates). We have:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \end{cases}$$

and note that r is the distance of the point (x,y) from the origin, therefore when $(x,y) \to 0$ we have $r \to 0$ and vice versa. So:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim_{r\to 0} \frac{(r\cos\theta)^2(r\sin\theta)}{r^2} = \lim_{r\to 0} r \underbrace{\cos\theta^2\sin\theta}_{\text{bounded}} = 0$$

Method 2. Since $0 \le x^2 \le x^2+y^2$, we have $0 \le \dfrac{x^2}{x^2+y^2} \le 1$, in other words, the term $\dfrac{x^2}{x^2+y^2}$ is bounded. Then:

$$0 \le \left|\frac{x^2y}{x^2+y^2} - 0\right| = \left|\frac{x^2y}{x^2+y^2}\right| = \frac{x^2|y|}{x^2+y^2} = \underbrace{\left(\frac{x^2}{x^2+y^2}\right)}_{\text{is bounded}} \underbrace{|y|}_{\text{tends to zero}} = 0$$

Then by the Sandwich Theorem, we have $\lim\limits_{(x,y)\to(0,0)} \left|\dfrac{x^2y}{x^2+y^2} - 0\right| = 0$. Equivalently we have:
$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2} = 0$

Example . Evaluate the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2\sin^2 y}{x^2+2y^2}$ or show that it does not exist.

Solution .

We substitute y for $\sin y$ as $y \to 0$, and therefore we study the following limit instead

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+2y^2}$$

Method 1 (by changing to polar coordinates).

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+2y^2} = \lim_{r\to 0} \frac{(r^2\cos^2\theta)(r^2\sin^2\theta)}{(r^2\cos^2\theta)+2(r^2\sin^2\theta)} = \lim_{r\to 0} r^2 \frac{\cos^2\theta\sin^2\theta}{\cos^2\theta+2\sin^2\theta}$$

$= \lim_{r\to 0} r^2 \left(\dfrac{\cos^2\theta}{\cos^2\theta+2\sin^2\theta}\right) \sin^2\theta =$ **see the next line**

But note that the quotient $\frac{\cos^2\theta}{\cos^2\theta+2\sin^2\theta}$ is bounded as the numerator is less than the denominator. On the other hand, the term $\sin^2\theta$ is bounded, hence the product $\left(\frac{\cos^2\theta}{\cos^2\theta+2\sin^2\theta}\right)\sin^2\theta$ is bounded. So, we continue:

$$= \lim_{r\to 0} \underbrace{r^2}_{\text{tends to zero}} \underbrace{\frac{\cos^2\theta}{\cos^2\theta+2\sin^2\theta}\sin^2\theta}_{\text{is bounded}}$$

Method 2. Since $0 \le x^2 \le x^2+2y^2$, we have $0 \le \dfrac{x^2}{x^2+2y^2} \le 1$, in other words, the term $\dfrac{x^2}{x^2+2y^2}$ is bounded. Then:

$$0 \le \left|\frac{x^2y^2}{x^2+2y^2} - 0\right| = \left|\frac{x^2y^2}{x^2+2y^2}\right| = \frac{x^2y^2}{x^2+2y^2} = \underbrace{\left(\frac{x^2}{x^2+2y^2}\right)}_{\text{is bounded}} \underbrace{y^2}_{\text{tends to zero}} = 0$$

Then by the Sandwich Theorem, we have $\displaystyle\lim_{(x,y)\to(0,0)} \left|\frac{x^2y^2}{x^2+2y^2} - 0\right| = 0$. Equivalently we have:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+2y^2} = 0$$