Equation of tangent plane:

for implicitly defined surfaces

section 12.9

Some surfaces are defined implicitly, such as the sphere $x^2 + y^2 + z^2 = 1$. In general an implicitly defined surface has the equation $F(x, y, z) = 0$. Consider a point $P = (x_0, y_0, z_0)$ on the surface. Suppose that the surface has a tangent plane at the point P. The tangent plane cannot be at the same time perpendicular to tree plane xy, xz, and yz. Without loss of generality assume that the tangent plane is not perpendicular to the xy-plane. Now consider two lines L_1 and L_2 on the tangent plane. Draw a plane π_1 through the line L_1 and perpendicular to the xy-plane. The plane π_1 cuts a curve C_1 out of the surface. The curve C_1 is through the point P. If $r(t) = \langle x(t)\ ,\ y(t)\ ,\ z(t)\rangle$ is parametrization for the curve C_1 with $r(t_0) = P$, then since the points of C_1 are on the surface, we have $F(x(t), y(t), z(t)) = 0$. Differentiating with respect to the parameter t gives:

$$
\begin{aligned}
0 &= \tfrac{d}{dt}F(x(t), y(t), z(t)) \\
&= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \\
&= F_x(x(t), y(t), z(t))\,x'(t) + F_y(x(t), y(t), z(t))\,y'(t) + F_z(x(t), y(t), z(t))\,z'(t)
\end{aligned}
$$

This shows that the vector $\nabla F(x(t), y(t), z(t))$ is perpendicular to the vector $r'(t) = \langle x'(t)\ ,\ y'(t)\ ,\ z'(t)\rangle$. Especially, at $t = t_0$. we will have that the vector ∇F at P is perpendicular to $r'(t_0)$. But, the vector $r'(t_0)$ is tangent to the curve C_1 and therefore is on the line L_1 as the line L_1 is tangent to C_1. So then, $\nabla F(P)$ is perpendicular to L_1.

Similarly one can show that $\nabla F(P)$ is perpendicular to L_2. So, the vector $\nabla F(P)$ is perpendicular to two lines on the plane, therefore it must be perpendicular to the plane.

tangent plane at (a,b,c) $\boxed{F_x(a,b,c)(x-a)+F_y(a,b,c)(y-b)+F_z(a,b,c)(z-c)=0}$

Since the gradient vector is perpendicular to the tangent plane, we can say that:

Theorem. The gradient vector is perpendicular to any point of an implicitly defined surface $F(x,y,z)=0$.

This equation shows that the vector $\langle F_z\,, F_y\,, F_z\rangle$ is the normal vector of the tangent plane.

Note. If $f(x,y)=k$ is the equation of a curve in the plane xy, then similarly one can show that the equation of the tangent line at (a,b) is:

$$f_x(a,b)(x-a)+f_y(a,b)(y-b)=0$$

Example. Find the equation of the the tangent line to the ellipse $x^2+2y^2=6$ at the point $(2,1)$.

Solution. Set $f(x,y)=x^2+2y^2$. Then:

$$\begin{cases} f_x=2x \\ f_y=4y \end{cases} \quad\Rightarrow\quad \begin{cases} f_x(2,1)=4 \\ f_y(2,1)=4 \end{cases}$$

Then the equation of the tangent plane will be:

$$4(x-2)+4(y-1)=0 \quad\Rightarrow\quad x+y=3$$

Example . Find the equation of the the tangent line to the ellipsoid $\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$ at the point $(-2,1,-3)$.

Solution. Set $F(x,y,z)=\frac{x^2}{4}+y^2+\frac{z^2}{9}$. So, the surface is implicitly written as $F(x,y,z)=3$

$$\begin{cases} F_x = \frac{x}{2} \\ F_y = 2y \\ F_z = \frac{2z}{9} \end{cases}$$

These derivatives at the point $(-2,1,-3)$ become:

$$\begin{cases} F_x = -1 \\ F_y = 2 \\ F_z = -\frac{2}{3} \end{cases}$$

Then the equation of the tangent plane:

$$-(x+2)+2(y-1)-\frac{2}{3}(z+3)=0 \quad \Rightarrow \quad 3x-6y+2z+18=0$$

And the equation of the normal line:

$$\frac{x+2}{-1}=\frac{y-1}{2}=\frac{z+3}{-\frac{2}{3}}$$

Example. The ellipsoid $4x^2+2y^2+z^2=15$ intersects the plane $y=2$ at an ellipse. Find the parametric equations of the tangent line to the ellipse at the point $(1,2,2)$.

Solution.

$$\begin{cases} F_x = 8x \\ F_y = 4y \\ F_z = 2z \end{cases} \quad \Rightarrow \quad \begin{cases} F_x(1,2,2) = 8 \\ F_y(1,2,2) = 8 \\ F_z(1,2,2) = 4 \end{cases}$$

Then the equation of the tangent plane:

$$8(x-1)+8(y-2)+4(z-2)=0 \quad \Rightarrow \quad 2(x-1)+2(y-2)+(z-2)=0$$

We then set $y=2$ in the equation:

$2(x-1)+(z-2)=0 \quad \Rightarrow \quad z=4-2x$

$$\begin{cases} x=x \\ y=2 \qquad\qquad -\infty<x<\infty \\ z=4-2x \end{cases}$$

We note further that the vector $\langle 1\,,0\,,-2\rangle$ is the direction vector of this line.

Example. At what points of the paraboloid $y=x^2+z^2$ the tangent plane is parallel to plane $x+2y+3z=1$?

Solution. Suppose that the point we are looking for is the point (a,b,c). We write the surface in the implicit form $F(x,y,z)=x^2+z^2-y=0$. Then:

$$\begin{cases} F_x=2x \\ F_y=-1 \\ F_z=2z \end{cases} \quad \Rightarrow \quad \begin{cases} F_x(a,b,c)=2a \\ F_y(a,b,c)=-1 \\ F_z(a,b,c)=2c \end{cases}$$

The gradient vector $\nabla F(a,b,c)$ is perpendicular to the tangent plane at (a,b,c) , therefore it must be parallel to the vector $\langle 1,2,3\rangle$, therefore it must satisfy:

$$\frac{2a}{1}=\frac{-1}{2}=\frac{2c}{3} \quad \Rightarrow \quad \begin{cases} 4a=-1 \\ 4c=-3 \end{cases} \quad \Rightarrow \quad \begin{cases} a=-\frac{1}{4} \\ c=-\frac{3}{4} \end{cases}$$

As the point (a,b,c) must be on the surface, we have:

$$b=a^2+c^2 \quad \Rightarrow \quad b=\frac{1}{16}+\frac{9}{16}=\frac{5}{8}$$

So

$$(a,b,c)=(-\frac{1}{4},\frac{5}{8},-\frac{3}{4})$$

Example . Show that the ellipsoid $3x^2+2y^2+z^2=9$ and the sphere

$x^2+y^2+z^2-8x-6y-8z+24=0$ are tangent to each other at the point $(1,1,2)$. (This means that they have a common tangent plane at the point.)

Solution. We consider the surfaces in the implicit form:

$$\begin{cases} F(x,y,z)=3x^2+2y^2+z^2=9 \\ G(x,y,z)=x^2+y^2+z^2-8x-6y-8z+24=0 \end{cases}$$

We must show that the surfaces have a common tangent plane at the point $(1,1,2)$. Equivalently, we show that the vectors $\nabla F(1,1,2)$ and $\nabla G(1,1,2)$ are parallel (because these two vectors are the normal vectors of those planes).

$$\begin{cases} F_x=6x \\ F_y=4y \\ F_z=2z \end{cases} \Rightarrow \begin{cases} F_x(1,1,2)=6 \\ F_y(1,1,2)=4 \\ F_z(1,1,2)=4 \end{cases}$$

$$\begin{cases} G_x=2x-8 \\ G_y=2y-6 \\ G_z=2z-8 \end{cases} \Rightarrow \begin{cases} G_x(1,1,2)=-6 \\ G_y(1,1,2)=-4 \\ G_z(1,1,2)=-4 \end{cases}$$

So we have $\begin{cases} \nabla F(1,1,2)=\langle 6,4,4\rangle \\ \nabla G(1,1,2)=\langle -6,-4,-4\rangle \end{cases}$ and obviously these two vectors are parallel.

Example. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z=x^2+y^2$ and the ellipsoid $4x^2+y^2+z^2=9$ at the point $(-1,1,2)$.

Solution. We write the two surfaces in the implicit form:

$$\begin{cases} F(x,y,z)=x^2+y^2-z=0 \\ G(x,y,z)=4x^2+y^2+z^2-9=0 \end{cases}$$

The tangent line we are looking for in the intersection of the tangent planes of the two surfaces. The vectors $\nabla F(-1, 1, 2)$ and $\nabla G(-1, 1, 2)$ are perpendicular to the surfaces at the common point $(-1, 1, 2)$ of the two surfaces. Therefore, the vector $\nabla F(-1, 1, 2) \times \nabla G(-1, 1, 2)$ is parallel to the tangent line, so we will use this vector to write the equation of the tangent line.

$$\begin{cases} F_x = 2x \\ F_y = 2y \\ F_z = -1 \end{cases} \Rightarrow \begin{cases} F_x(-1,1,2) = -2 \\ F_y(-1,1,2) = 2 \\ F_z(-1,1,2) = -1 \end{cases}$$

$$\begin{cases} G_x = 8x \\ G_y = 2y \\ G_z = 2z \end{cases} \Rightarrow \begin{cases} G_x(-1,1,2) = -8 \\ G_y(-1,1,2) = 2 \\ G_z(-1,1,2) = 4 \end{cases}$$

$$\nabla F(-1, 1, 2) \times \nabla G(-1, 1, 2) = \begin{vmatrix} i & j & k \\ -2 & 2 & -1 \\ -8 & 2 & 4 \end{vmatrix} = \langle 10, 16, 12 \rangle$$

The vector $\langle 10, 16, 12 \rangle$ is the direction vector of the tangent line. Therefore its parametric equations are:

$$\begin{cases} x = 5t - 1 \\ y = 8t + 1 \\ z = 6t + 2 \end{cases} \qquad -\infty < t < \infty$$