Related Rates
section 3.9

Important Note: In solving the related rates problems , the rate of change of a quantity is given and the rate of change of another quantity is asked for. You need to find a relationship between the two quantities to be able to solve the problem.

Example (from the textbook): Air is being pumped into a sphere balloon so that its volume increases at a rate of 100 $\frac{\text{cm}^3}{\text{s}}$. How fast is the radius of the balloon is increasing when the diameter is 50 cm ?.

Solution.

We are given that:

$$\frac{dV}{dt} = 100 \ \frac{\text{cm}^3}{\text{s}}$$

We want to find :

$$\frac{dr}{dt} \qquad \text{when} \quad r = 25 \text{ cm}$$

So, here is what we do for it:

$$V = \frac{4}{3}\pi r^3 \qquad \Rightarrow$$

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \Rightarrow \qquad \textbf{substitute} \qquad \Rightarrow$$

$$100 = 4\pi(25)^2 \frac{dr}{dt} \qquad \Rightarrow \qquad \frac{dr}{dt} = \frac{100}{4\pi(25)^2} = \frac{1}{25\,\pi} \ \frac{\text{cm}}{\text{s}}$$

and this value is positive, meaning that the radius is increasing, as expected.

Example : Air is being pumped out of a sphere balloon so that its volume decreases at a rate of 50 $\frac{\text{cm}^3}{\text{s}}$. How fast is the radius of the balloon is decreasing when the diameter is 10 cm ?.

Solution.

We are given that:

$$\frac{dV}{dt} = -50\,\frac{\text{cm}^3}{\text{s}} \qquad \textbf{be careful about the negative sign}$$

We want to find :

$$\frac{dr}{dt} \qquad \text{when} \;\; r = 5 \text{ cm}$$

So, here is what we do for it:

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow$$

$$\frac{dV}{dt} = \frac{dV}{dr}\,\frac{dr}{dt} = 4\pi r^2\,\frac{dr}{dt} \quad \Rightarrow \quad \textbf{substitute} \quad \Rightarrow$$

$$-50 = 4\pi(5)^2\,\frac{dr}{dt} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{-50}{4\pi(5)^2} = \frac{-1}{2\,\pi}\;\frac{\text{cm}}{\text{s}}$$

and this value is negative, meaning that the radius is decreasing, as expected.

Example : The area of a square is decreasing at the rate 1 $\frac{\mathrm{m}^2}{s}$ when its side is 570 cm. Find the rate of change of its perimeter at that moment; describe it in terms of $\frac{\mathrm{m}}{\mathrm{min}}$.

Solution. Let us denote the side by x , the area by A , and the perimeter by P. So:

$$\begin{cases} P = 4x \\ A = x^2 \end{cases}$$

Along the path , we change all quantities of length to meters and all quantities of time to minutes. Then

$$A = x^2 \quad \Rightarrow \quad \frac{dA}{dt} = 2x\,\frac{dx}{dt} \qquad\qquad (1)$$

For a particular moment we are given

$$\left.\begin{array}{l} \frac{dA}{dt} = -1\ \frac{\mathrm{m}^2}{\mathrm{s}} = (-1)(60)\ \frac{\mathrm{m}^2}{\mathrm{min}} \\ \\ x = 570\ \mathrm{cm}\ =\ 5.7\ \mathrm{m} \end{array}\right\} \quad \overset{(1)}{\Rightarrow} \quad -60 = 2(5.7)\,\frac{dx}{dt} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{-60}{11.4}\ \frac{\mathrm{m}}{\mathrm{min}}$$

Next Step:

$$P = 4x \quad \Rightarrow \quad \frac{dP}{dt} = 4\,\frac{dx}{dt} \quad \Rightarrow \quad \frac{dP}{dt} = (4)\left(\frac{-60}{11.4}\right) = \frac{-240}{11.4}\ \frac{\mathrm{m}}{\mathrm{min}} \qquad \checkmark$$

Important note 1. Suppose that $f(t)$ is a function of time t . When we say

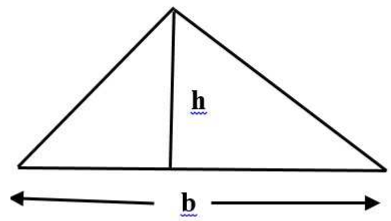
$$f \text{ in decreasing at the rate of 0.3 per seconds at time } t = t_0$$

we mean that the derivative of f with respect to t at time t_0 is -0.3 , i.e. $f'(t_0) = -0.3$. and , when we say that f is increasing at the rate of 1.7 at time $t = t_0$, we mean that $f'(t_0) = 1.7$.

Important note 2. Suppose that $f(t)$ is a function of time t and that the unit of measurement for the values $f(t)$ is meters. Suppose that the unit of measurement for t is seconds. Then the values $f(t) - f(t_0)$ are also measured in meters , and the values $t - t_0$ are measured in seconds , therefore the unit of measurement for the quotients $\frac{f(t)-f(t_0)}{t-t_0}$ is $\frac{\text{meters}}{\text{second}}$, and then the unit of measurement for the limit $f'(t_0) = \lim_{t \to t_0} \frac{f(t)-f(t_0)}{t-t_0}$ is $\frac{\text{meters}}{\text{second}}$. And so on , ...

Important note 3. Use negative numbers to denote decreasing rates of change . See the following example for this note.

Example **(FALL 2016 - midterm exam)**. The altitude of a triangle is increasing at a rate of 1 $\frac{\text{cm}}{\text{min}}$, while the area is increasing at a rate of 2 $\frac{\text{cm}^2}{\text{min}}$ (see figure below). At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2



Solution.

Given: $\dfrac{dh}{dt} = 1 \quad \text{and} \quad \dfrac{dA}{dt} = 2$

$\dfrac{db}{dt}\,? \quad \text{when} \quad h = 10 \quad \text{and} \quad A = 100$

$A = \frac{1}{2}bh \quad \Rightarrow \quad \frac{dA}{dt} = \frac{1}{2}\frac{db}{dt}h + \frac{1}{2}b\,\frac{dh}{dt}$

Substitute $\frac{dA}{dt} = 2$ and $h = 10$ and $\frac{dh}{dt} = 1$ to get:

$2 = \frac{1}{2}\frac{db}{dt}(10) + \frac{1}{2}b\,(1) \quad \xRightarrow{\text{multiply by 2}} \quad 1 = 10\,\frac{db}{dt} + b \qquad (1)$

Now we need to find b in order to be able to find $\frac{db}{dt}$ at that moment.

$A = \frac{1}{2}bh \quad \xRightarrow{A=100\ ,\ h=10} \quad 100 = \frac{1}{2}b(10) \quad \Rightarrow \quad b = 20$ at that particular moment

Putting this value back into the equality (1) , we get:

$1 = 10\,\frac{db}{dt} + b \quad \Rightarrow \quad 1 = 10\,\frac{db}{dt} + 20 \quad \Rightarrow \quad \frac{db}{dt} = \frac{-19}{10} = -1.9 \;\; \frac{\text{cm}}{\text{min}}$

Example (FALL 2015 - midterm exam). Sand is poured into a conical pile at a rate of 20 m^3 per minute. The diameter of the cone is always equal to its height. How fast is the height of the conical pile increasing when the pile is 10 m high?.

Solution.

Given: $\dfrac{dV}{dt} = 20 \quad \text{and} \quad 2r = h$

$\dfrac{dh}{dt}\,? \quad \text{when} \quad h = 10\ m$

$V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$

$\frac{dV}{dt} = \frac{\pi}{12}(3h^2)\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$

$20 = \frac{\pi}{4}(10)^2\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{4}{5\pi}\ \frac{\text{m}}{\text{min}}$

More questions will be added to this file