Relative Maxima and Minima
sections 4.3

<u>**Definition**</u>. By a critical point of a function f we mean a point x_0 in the domain at which either the derivative is zero or it does not exists. So , geometrically , one of the following cases happen at a critical point:

(i) the tangent line is horizontal

(ii) the tangent line is vertical (this corresponds to the case where the derivative is one of $\pm\infty$)

(iii) the tangent line does not exist ; there is a cusp on the graph at x_0

<u>**Example**</u>. Find the critical points of the function $f(x) = \frac{x^2}{x^3-1}$

<u>**Solution**</u>.

$$f'(x) = \frac{(x^2)'(x^3-1) - (x^2)(x^3-1)'}{(x^3-1)^2} = \cdots\cdots = \frac{-x(x^3+2)}{(x^3-1)^2}$$

Now we need to look for the points at which the derivative is zero or the derivative does not exist.

$$f'(x) = 0 \qquad \Rightarrow \qquad x = 0 \ , \ -\sqrt[3]{2}$$

there is no point in the domain where f is not differentiable (note that the point $x = 1$ is not in the domain at all , so it is not considered as a critical point). So , the only critical points are $x = 0$, $-\sqrt[3]{2}$.

<u>**Definition**</u>. A function f is said to have **relative minimum** (**local minimum**) at x_0 if there exists some open interval I containing x_0 such that

$$f(x_0) \leq f(x) \qquad \text{for all } x \in I$$

<u>**Definition**</u>. A function f is said to have **relative maximum** (**local maximum**) at x_0 if there exists some open interval I containing x_0 such that

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Question. The following theorem tells us where to search for relative extrema ; in fact it says that we need to llok for the critical points if we really want to search for the relative extrema.

Theorem. If a function f has a relative extrema over an interval I at a point $x_0 \in I$ and if x_0 is not an endpoint of I , then x_0 is a critical point of f , and therefore either $f(x_0) = 0$ or $f'(x_0)$ does not exist.

First-Derivative Test for Relative Extrema. Suppose that c is a critical point of a continuous function f ($f'(c)$ may or may not exist).

(i) If on both sides of c we have this situation:

		c
f'	$-$	$+$
f	$\searrow$	$\nearrow$

then f has a relative minimum at c .

(ii) If on both sides of c we have this situation:

		c
f'	$+$	$-$
f	$\nearrow$	$\searrow$

then f has a relative maximum at c .

(iii) If on both sides of c the derivative f' does not change sign:

		c
f'	$+$	$+$
f	$\nearrow$	$\nearrow$

		c
f'	$-$	$-$
f	$\searrow$	$\searrow$

then f has neither a relative max or relative min at c .

Example (from the textbook). Find the relative maximums and relative minimums of the function $f(x) = x^{\frac{5}{3}} - x^{\frac{2}{3}} = \sqrt[3]{x^5} - \sqrt[3]{x^2}$

Solution.

$$f'(x) \;=\; \frac{5}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}} \;=\; \frac{5}{3}\sqrt[3]{x^2} - \frac{2}{3\sqrt[3]{x}} \;=\; \frac{5(\sqrt[3]{x^2})(\sqrt[3]{x}) - 2}{3\sqrt[3]{x}} \;=\; \frac{5x-2}{3\sqrt[3]{x}}$$

$$f'(x) = 0 \quad \Rightarrow \quad 5x - 2 = 0 \quad \Rightarrow \quad x = \frac{2}{5} \quad \text{(a critical point)}$$

The function is not differentiable at the point $x = 0$ of the domain (another critical point). So we have only two critical points: $x = 0$ and $x = \frac{2}{5}$

		0		$\mathbf{\frac{2}{5}}$	
$\sqrt[3]{x}$	−	•	+		+
5x − 2	−		−	•	+
f′	+		−		+
f	↗	**relative max**	↘	**relative min**	↗

Second-Derivative Test for Relative Extrema. Suppose $f''(x)$ is continuous on an interval containing a point c and that c is a critical point of type $f'(c) = 0$. Then

- If $f''(c) > 0$, then c is a point of relative minimum for f .
- If $f''(c) < 0$, then c is a point of relative maximum for f .
- If $f''(c) = 0$, then no conclusion can be made about c .

Note. The first-derivative test is used for both types of critical points no matter whether $f'(c) = 0$ or $f'(c)$ does not exist . But , the second-derivative test is used only for the critical points satisfying $f'(c) = 0$. For example, in the previous example , the second-derivative test cannot be applied for the critical point $x = 0$. But , it can be used to decide about $x = \frac{2}{5}$.

$$f'(x) = \frac{1}{3}(5x-2)x^{-\frac{1}{3}} \quad \Rightarrow$$

$$f''(x) = \frac{1}{3}(5x-2)'(x^{-\frac{1}{3}}) + \frac{1}{3}(5x-2)(x^{-\frac{1}{3}})'$$

$$= \frac{1}{3}(5)(x^{-\frac{1}{3}}) + \frac{1}{3}(5x-2)(-\frac{1}{3}x^{-\frac{4}{3}}) = \frac{5}{3\sqrt[3]{x}} - \frac{5x-2}{9x\sqrt[3]{x}}$$

$$= \frac{15x-(5x-2)}{9x\sqrt[3]{x}} = \frac{10x+2}{9x\sqrt[3]{x}}$$

$$\Rightarrow \quad f''(\frac{2}{5}) > 0 \quad \Rightarrow \quad \frac{2}{5} \text{ gives relative minimum}$$

Example (section 4.3 exercise 19). Find the the critical points of the function $f(x) = (x+2)^3(x-4)^3$, and determine which critical points give relative maxima and which ones give relative minima.

<u>Solution</u>.

$$f'(x) = \{(x+2)^3\}'\{(x-4)^3\} + \{(x+2)^3\}\{(x-4)^3\}'$$

$$= \{3(x+2)^2\}\{(x-4)^3\} + \{(x+2)^3\}\{3(x-4)^2\}$$

$$= 3(x+2)^2(x-4)^2\{(x-4)+(x+2)\}$$

$$= 3(x+2)^2(x-4)^2\{2x-2\}$$

$$= 6(x+2)^2(x-4)^2(x-1)$$

$f'(x) = 0 \quad \Rightarrow \quad x = -2\,,\, 1\,,\, 4 \quad$ (the only critical points)

(there are no critical points at which f' does not exist).

The only term whose sign must be determined is the term $(x-1)$ because the signs of the other terms don't change.

		-2	1	4
x − 1	−	−	• +	+
f′	−	−	+	+
f	↘	↘	↗	↗
			relative min	

Note. The second-derivative test cannot decide about the critical points -2 and 4 because at these points we have $f''(x) = 0$. But for the critical point $x = 1$ we can use that test:

$$f''(x) = 6\{(x+2)^2\}'\{(x-4)^2\}\{(x-1)\}+6\{(x+2)^2\}\{(x-4)^2\}'\{(x-1)\}+6\{(x+2)^2\}\{(x-4)^2\}\{(x-1)\}'$$

$$= 6\{2(x+2)\}\{(x-4)^2\}\{(x-1)\} + 6\{(x+2)^2\}\{2(x-4)\}\{(x-1)\} + 6\{(x+2)^2\}\{(x-4)^2\}\{1\}+$$

$$= 6(x+2)(x-4)\{2(x^2 - 5x + 4) + 2(x^2 + x - 2) + (x^2 - 2x - 8)\}$$

$$= \cdots\cdots = 6(x+2)(x-4)\{5x^2 - 10x - 4\}$$

$$\Rightarrow \quad f''(1) = (6)(3)(-3)(-9) > 0 \quad \Rightarrow \quad x = 1 \text{ is a relative minimum}$$

Note. It seems that the first-derivative test is easier and more comprehensive than the second-derivative test , therefore in your exam use the first-derivative test.

Example. Find the critical points of the function $f(x) = 3x^4 + 8x^3 + 6x^2$ and determine which ones give relative maxima and which ones give relative minima.

Solution.

$f'(x) = 12x^3 + 24x^2 + 12x = 12x(x^2 + 2x + 1)$

To factorize $x^2 + 2x + 1$ into linear forms , we do this:

$x^2 + 2x + 1 = 0 \quad \Rightarrow \quad x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2\pm\sqrt{4-4}}{2} = \frac{-2+\pm 0}{2} = -1$ (**multiple root**)

So then

$f'(x) = 12x(x-1)^2$

We only need to determine the sign of x because the sign of the other term does not change.

		-1		0	
$\mathbf{x}$	$-$		$-$	$\bullet$	$+$
$\mathbf{f'}$	$-$		$-$		$+$
$\mathbf{f}$	↘		↘	**relative minimum**	↗