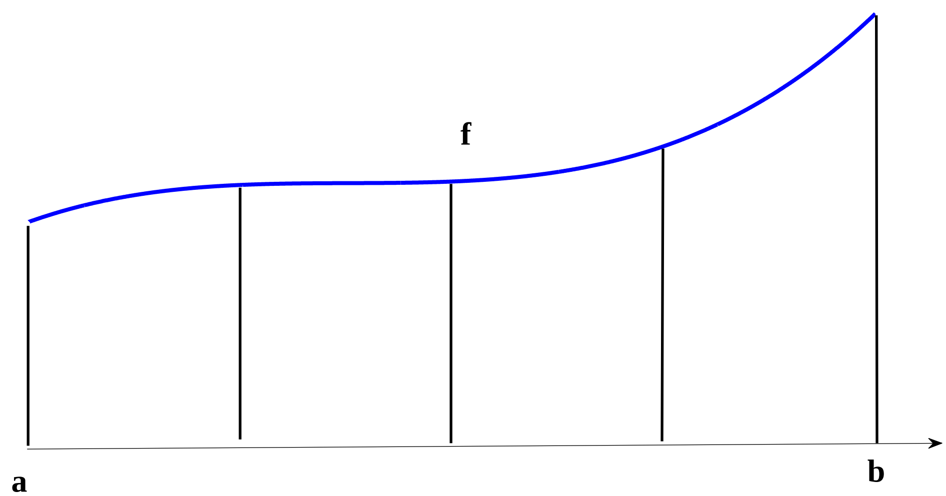
The Definite Integral
Section 6.3

Consider the graph of a function f over a closed interval $[a, b]$. Divide the interval $[a, b]$ into n subintervals of equal length $\frac{b-a}{n}$. In the figure below we have divided the interval $[a, b]$ into 4 subintervals each of length $\frac{b-a}{4}$.



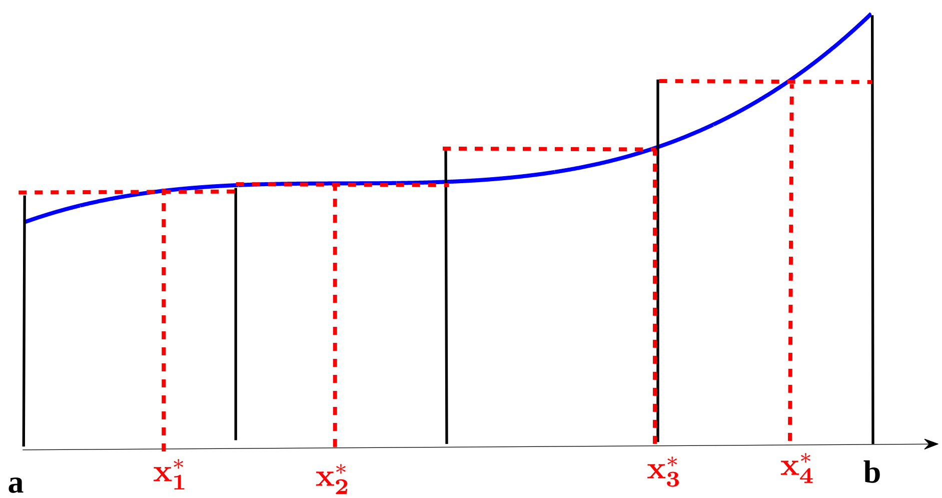
Now in each subinterval choose a point x^* (arbitrarily anywhere in that interval , it can be one of the endpoints too); so in the first interval we choose x_1^* in the second interval we choose x_2^* , and so on. Note that , for example , we have chosen x_3^* to be one of the endpoints of the subinterval it belongs to.

Then above each of these x^* points draw a vertical line extending from the x-axis to where the graph is. The line segment is actually the value of f at the point x^*. Then form a rectangle having the height $f(x^*)$ and base equal to $\frac{b-a}{4}$. The sum of the areas of the rectangles such found is an approximation for the area of the region under the graph. This sum is equal to

$$f(x_1^*)\frac{b-a}{4} + f(x_2^*)\frac{b-a}{4} + f(x_3^*)\frac{b-a}{4} + f(x_4^*)\frac{b-a}{4}$$

which is actually equal to

$$\frac{b-a}{4}\Big\{f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*)\Big\}$$



In order to have a better approximation we might need to "refine" this partitioning of the interval $[a, b]$, so as to have more and more subintervals. So, in general , let us divide the interval $[a, b]$ into n subintervals with equal lengths $\frac{b-a}{n}$, and then from each subinterval choose an arbitrary point x^* and do the above process with these points. Then we come up with some rectangles which together approximate the area under the graph and above the x-axis. The sum of areas of the rectangles , as you guess, is

$$\frac{b-a}{n}\Big\{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)\Big\} = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i^*)$$

As we said, the approximation gets better and better if we have a large number of subintervals , i.e. if n is large , in other words if $n \to \infty$. If the function f is "nice enough" then the limit

$$\lim_{n\to\infty} \frac{b-a}{n}\sum_{i=1}^{n} f(x_i^*)$$

tends to the area under the graph of f . If this limits exists , then we say that function f is integrable over the interval $[a, b]$, in which case the value of the limit is denoted by $\int_a^b f(x)dx$ or $\int_a^b f(t)dt$, and so on :

$$\int_a^b f(x)dx \overset{\textbf{definition}}{=} \lim_{n\to\infty} \frac{b-a}{n}\sum_{i=1}^{n} f(x_i^*)$$

Here is an important question : What functions are nice in this study ?. In other words , for what functions does this limit exist?. The following theorem gives a good answer:

Theorem. If f is continuous on $[a, b]$, then the integral $\int_a^b f(x)dx$ exists , i.e. the above limit exist.

Note. In practice , we do not calculate this limit to find the area of the region under the graph of f , because it is not practical due to its complexity. Instead , we use a very powerful tool called "The First Fundamental Theorem Of Calculus" ; we will study this theorem in the next lecture.