Theorem (transmission property of continuous functions): If f is continuous at L and if $\lim_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f(g(x)) = f(L)$$

equivalently

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

Note: This theorem is true for the limits $\lim_{x\to a^+}$, $\lim_{x\to a^-}$, $\lim_{x\to\infty}$, and $\lim_{x\to-\infty}$ too.

Example: Evaluate the limit $\lim_{x\to-\infty} \sin(\sqrt{x^2+x+1} - \sqrt{x^2-x+1})$.

Solution:

$$\lim_{x\to-\infty} \sin(\sqrt{x^2+x+1} - \sqrt{x^2-x+1}) = \sin\left(\lim_{x\to-\infty}(\sqrt{x^2+x+1} - \sqrt{x^2+x-1})\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{(\sqrt{x^2+x+1} - \sqrt{x^2-x+1})(\sqrt{x^2+x+1} + \sqrt{x^2-x+1})}{(\sqrt{x^2+x+1} + \sqrt{x^2-x+1})}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{(x^2+x+1) - (x^2-x+1)}{\sqrt{x^2+x+1} + \sqrt{x^2-x+1}}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{2x}{\sqrt{x^2+x+1} + \sqrt{x^2-x+1}}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{2x}{\sqrt{x^2(1+\frac{1}{x}+\frac{1}{x^2})} + \sqrt{x^2(1-\frac{1}{x}+\frac{1}{x^2})}}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{2x}{|x|\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} + |x|\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{2x}{(-x)\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+(-x)\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{2}{(-1)\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+(-1)\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}}\right)$$

$$= \sin\left(\lim_{x\to-\infty} \frac{2}{(-1)\sqrt{1+0+0}+(-1)\sqrt{1-0+0}}\right) = \sin(-1) \qquad \checkmark$$