Definition: By a function we mean a rule $y = f(x)$ that assigns only one value y to each **allowable** value x. The graph of a function intersects a vertical line in at most one point and not more. The x is called the **independent variable**, and the y is called the **dependent variable**.

Example: The equation $x^2+y^2 = 1$ defines the unit circle but this equation does not represent a function as some vertical lines intersect the unit circle in more than one point.

Definition: The set of allowable x's at which the function is define is called the domain of the function. The set of all values $f(x)$ of the function is called the **range** of the function f. Graphically the domain is found by projecting the points of the graph onto the x-axis and the range is found by projecting the graph onto the y-axis.

Example: The function

$$f(x) = x - 1 \qquad\qquad x \geq 3$$

the value $f(-1)$ is not defined since the point $x = -1$ is not in the domain. For this example the world of x's which we are allowed to work with is the interval $[3\,,\,\infty)$

Example: Although The functions

$$f_1(x) = x^2 - 1$$
$$f_2(x) = x^2 - 1 \qquad 1 \leq x < 5$$

look similar, but they are not equal because they have different domains.

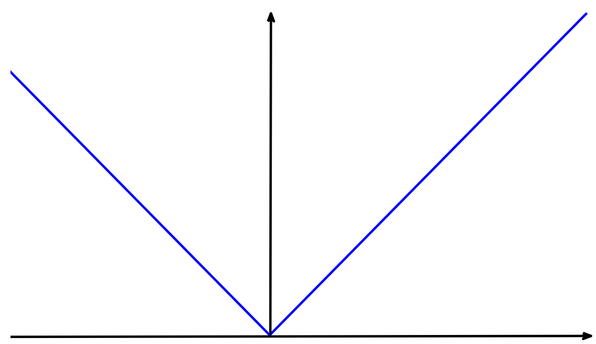
Note: If $f(x)$ is the rule defining a function and if (a, b) is on the graph of the function, then $b = f(a)$. For example, for the function $f(x) = x^2$, we have $f(-2) = 4$ and that the point $(-2, 4)$ is on the graph.

The absolute value $|x|$ of any number x is defined by

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

$$|2.7| = 2.7 \qquad\qquad |-3.5| = 3.5$$

The graph of the function $f(x) = |x|$:



Facts you should know:

$$\boxed{\sqrt{x^2} = |x| \qquad\qquad |x|^2 = x^2 \qquad\qquad |-x| = |x|}$$

For example

$$\sqrt{(-3)^2} = \sqrt{9} = 3 = |-3| \qquad \Rightarrow \qquad \sqrt{x^2} = |x| \quad \text{for} \quad x = -3$$

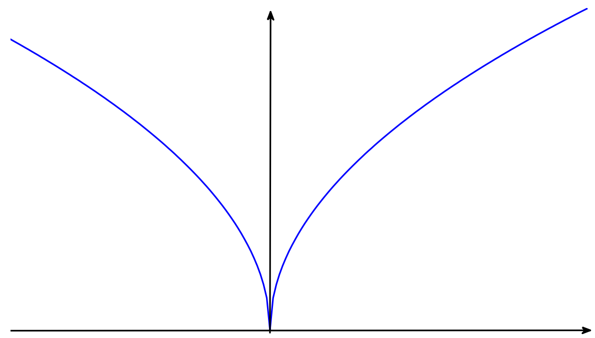
Example: Draw the graph of the function

$$y + 1 = |2x - 1|$$

The graphs of the functions

$$y = x^2$$
$$y = \sqrt{|x|}$$

are symmetric about the y-axis:



Such functions are called **<u>even</u>**. Note that for such functions the domain is symmetric about the y-axis and that in the domain we have $f(-x) = f(x)$.

The graph of the function $y = x^5 - x$ is symmetric with respect to the origin. Such a function is called an **<u>odd function</u>**. Note that for such a function the domain is symmetric with respect to the origin and that in the domain we have $f(-x) = -f(x)$. So:

Definition. A function is called **<u>even</u>** if its domain is symmetric with respect to the origin and has this property:

$$f(-x) = f(x)$$

A function is called **<u>odd</u>** if its domain is symmetric with respect to the origin and satisfies:

$$f(-x) = -f(x)$$

• Here is the way we algebraically verify that a function is even or odd or neither:

Example. The function $f(x) = \sqrt{|x|}$ is even because:

$$f(-x) = \sqrt{|-x|} = \sqrt{|x|} = f(x)$$

Example. The function $f(x) = x^5 - x$ is an odd function because:

$$f(-x) = (-x)^5 - (-x) = -x^5 + x = -(x^5 - x) = -f(x)$$

Example. The function $f(x) = x^2 + x$ is neither odd nor even:

$$f(-x) = (-x)^2 + (-x) = x^2 - x$$

which is not equal to either of $f(x) = x^2 + x$ and $-f(x) = -x^2 - x$. Or you may check that its graph is neither symmetric with respect to the y-axis nor being symmetric with respect to the origin.

Note: We can write every function $f(x)$ as the sum of an even function and an odd function:

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd part}}$$

The textbook uses the symbols f_e and f_o to refer to the even and odd parts of a function f :

$$f_e(x) = \frac{f(x) + f(-x)}{2} \qquad\qquad f_o(x) = \frac{f(x) - f(-x)}{2}$$

Example. Write the function $g(x) = \frac{x+1}{x-2}$ in terms of its even and odd parts.

Solution.

$$g_e(x) = \frac{g(x) + g(-x)}{2} = \frac{1}{2}\left\{\frac{x+1}{x-2} + \frac{-x+1}{-x-2}\right\} = \frac{1}{2}\left\{\frac{x+1}{x-2} - \frac{-x+1}{x+2}\right\} = \frac{x^2+2}{x^2-4}$$

$$g_o(x) = \frac{g(x) - g(-x)}{2} = \frac{1}{2}\left\{\frac{x+1}{x-2} - \frac{-x+1}{-x-2}\right\} = \frac{1}{2}\left\{\frac{x+1}{x-2} + \frac{-x+1}{x+2}\right\} = \frac{3x}{x^2-4}$$

$$g(x) = \underbrace{\frac{x^2+2}{x^2-4}}_{even} + \underbrace{\frac{3x}{x^2-4}}_{\text{odd}}$$

A **polynomial** of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where a_n nd a_{n-1} , ... , a_1 , and a_0 are some constants. The domain of this function is the whole real line $\mathbb{R} = (-\infty\,,\,+\infty)$. For example, the function

$$p(x) = 2\,x^3 - \sqrt{3}\;x + 1$$

is a polynomial of degree 3 , and its domain is the real line.

By a **rational function** we mean a function of the form $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are some polynomials. The domain of such a function consists of those points x for which $Q(x) \neq 0$.

For example, the function

$$f(x) = \frac{x^3 + 2x^2 + x + 1}{x^2 - x - 1}$$

is a rational function. Since the roots of the denominator are $\frac{1 \pm \sqrt{5}}{2}$ the domain of the function f is

$$\mathbb{R} - \left\{ \frac{1 - \sqrt{5}}{2} \, , \, \frac{1 + \sqrt{5}}{2} \right\}$$