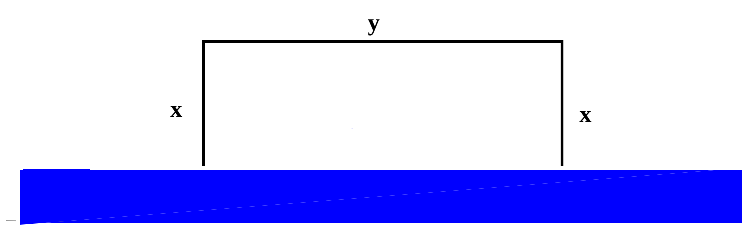
second part of section 4.7

Example (from the textbook). A rectangular field is to be fenced on three sides with 1000 m of fencing (the fourth side being a straight river edge). Find the dimensions of the field in order that the area be as large as possible.

Solution.



From the figure we have

$$\text{area} \quad A = xy$$

where $2x + y = 1000$. Using this equality $y = 1000 - 2x$ we can transform A into a function of one variable:

$$A(x) = x(1000 - 2x) = 1000x - 2x^2$$

Since x and y are positive numbers (edges are positive) , we are restricted to $0 < x < 500$ in order for $2x + y = 1000$ to hold. This is the domain of the function $A(x)$

$$A(x) = 1000x - 2x^2 \qquad 0 < x < 500$$

There is difference between this domain and the one that the textbook chooses; the textbook chooses the <u>closed</u> interval $0 \leq x \leq 500$ as the domain.

We want to maximize A. For this we differentiate the function:

$$A'(x) = 1000 - 4x$$

$$A'(x) = 0 \quad \Rightarrow \quad x = \frac{1000}{4} = 250$$

There are no points where $A'(x)$ does not exist , therefore the only critical point is $x = 250$ at which we have

$$A(250) = 125000$$

At the endpoints we calculate the limits:

$$\begin{cases} \lim_{x \to 0^+} A(x) = 0 \\ \lim_{x \to 500^-} A(x) = 0 \end{cases}$$

	candidate x	$f(x)$ or $\lim f(x)$	
	250	125000	**absolute max**
(not a candidate)	0	0	
(not a candidate)	500	0	

The following question is similar to exercise 62 of the list :

Section 4.7 exercise 23. Find the points on the hyperbola $y^2 - x^2 = 9$ closest to $(4, 0)$

Solution. The distance between two arbitrary points $(x_1\,,\,y_1)$ and $(x_2\,,\,y_2)$ is found through the formula

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Using this fact , if (x, y) an arbitrary point on the parabola , then its distance from the point $(4, 0)$ is

$$\sqrt{(x-4)^2 + (y-0)^2} = \sqrt{x^2 - 8x + 16 + y^2} \qquad (1)$$

We want to minimize this distance. But on the parabola we have $y^2 = 9 + x^2$,therefore the quatity (1) reduces to

$$\sqrt{2x^2 - 8x + 25}$$

But since we are looking for the ideal point (x, y) , we can equivalently minimize the square of this expression , which is:

$$f(x) = 2x^2 - 8x + 25 \qquad\qquad -\infty < x < \infty$$

Here is how: $f'(x) = 4x - 8$

$f'(x) = 0 \quad \Rightarrow \quad x = 2$

But:

$$\begin{cases} \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(2x^2 - 8x + 25) = +\infty \\ \\ \lim_{x\to\infty} f(x) = \lim_{x\to\infty}(2x^2 - 8x + 25) = +\infty \end{cases}$$

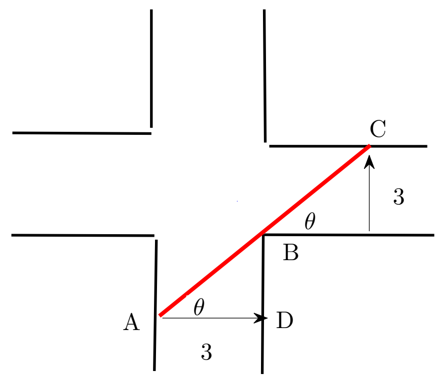
So we can form the following table:

	candidate x	$f(x)$ or $\lim f(x)$	
	2	17	**absolute min**
(not a candidate)	$-\infty$	$+\infty$	
(not a candidate)	∞	$+\infty$	

Note that for $x = 2$ we have

$y^2 = 9 + x^2 = 13 \quad \Rightarrow \quad y = \pm\sqrt{13} \quad \Rightarrow \quad (x, y) = (2\,,\,-\sqrt{13})\,,\,(2\,,\,\sqrt{13})$

Section 4.7 exercise 41 (modified). Two corridors , both 3 m wide, meet at right angles. Find the length of the longest beam that can be transported horizontally around the corner. Ignore the dimensions of the beam.



Solution. The length of longest transportable beam is the length of smallest line segment AC that can pass through the fixed point B at the corner (this was explained in class). But

(length of AC) $\quad \|AC\| = \|AB\| + \|BC\|$

But :

$\|AD\| = \|AB\| \cos\theta \quad \Rightarrow \quad \|AB\| = \|AD\| \sec\theta = 3\sec\theta$

Similarly,

$\|BC\| = 3 \csc\theta$

Then :

$\|AC\| = \|AB\| + \|BC\| = 3\sec\theta + 3\csc\theta \qquad\qquad 0 < \theta < \frac{\pi}{2}$

We need to minimize the length function

$f(\theta) = 3\sec\theta + 3\csc\theta \qquad\qquad 0 < \theta < \frac{\pi}{2}$

$f'(\theta) = 3\sec\theta\tan\theta - 3\csc\theta\cot\theta = \frac{3\sin\theta}{\cos^2\theta} - \frac{3\cos\theta}{\sin^2\theta}$

$f'(\theta) = 0 \quad \Rightarrow \quad \frac{3\sin\theta}{\cos^2\theta} = \frac{3\cos\theta}{\sin^2\theta} \quad \Rightarrow \quad \tan^3\theta = 1$

$\Rightarrow \quad \tan\theta = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{4}$ radians

For this value of θ we have $f(\frac{\pi}{4}) = 6\sqrt{2}$

But also:

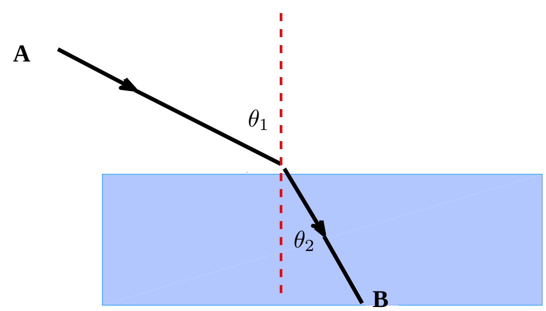
$$\begin{cases} \lim_{\theta\to 0^+} f(\theta) = \lim_{\theta\to 0^+} 3\sec\theta + 3\csc\theta = +\infty \\ \lim_{\theta\to (\frac{\pi}{2})^+} f(\theta) = \lim_{\theta\to 0^+} 3\sec\theta + 3\csc\theta = +\infty \end{cases}$$

	candidate θ	$f(\theta)$ or $\lim f(\theta)$	
	$\frac{\pi}{4}$	$6\sqrt{2}$	**absolute min**
(not a candidate)	0	∞	
(not a candidate)	$\frac{\pi}{2}$	∞	

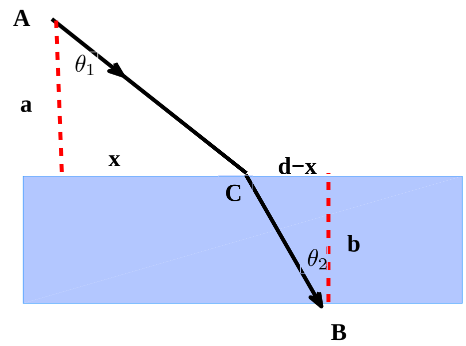
Therefore the length of the longest beam is $6\sqrt{2}$.

Example. Let v_1 and v_2 be the velocities of light in air and in water. A ray of light travels from the point A in the air to the point B in the water in such a way that the travel time is minimized. Using the figure shown below prove the Snell's Law:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$



Solution. Denote by d the horizontal distance between A and B. We denote the distance between A and the water surface by a, and denote the distance between B and the water surface by b.



travel time from A to $C = \frac{\|AC\|}{v_1} = \frac{\sqrt{a^2+x^2}}{v_1}$

travel time from C to $B = \frac{\|CB\|}{v_2} = \frac{\sqrt{(d-x)^2+b^2}}{v_2}$

Add up the two values to get the total travel time:

$$\begin{aligned} T(x) &= \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{(d-x)^2+b^2}}{v_2} \\ &= \frac{1}{v_1}(a^2+x^2)^{\frac{1}{2}} + \frac{1}{v_2}((d-x)^2+b^2)^{\frac{1}{2}} \qquad 0 \le x \le d \end{aligned}$$

Now differentiate:

$$\begin{aligned} T'(x) &= \frac{1}{2}\frac{1}{v_1}(a^2+x^2)^{-\frac{1}{2}}(2x) + \frac{1}{2}\frac{1}{v_2}\Big(b^2+(d-x)^2\Big)^{-\frac{1}{2}}\{-2(d-x)\} \\ &= \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}} \end{aligned}$$

$$T'(x) = 0 \quad \Rightarrow \quad \frac{x}{v_1\sqrt{a^2+x^2}} = \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}} \qquad (1)$$

But from the figure we see that

$$\sin\theta_1 = \frac{x}{\|AC\|} = \frac{x}{\sqrt{a^2+x^2}} \qquad\qquad \sin\theta_2 = \frac{d-x}{\|CB\|} = \frac{d-x}{\sqrt{b^2+(d-x)^2}}$$

Using these equalities , the equality (1) reduces to

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \qquad (2)$$

which is the required equality. It only remains to show that the point satisfying (1) , or satisfying the equivalent equality (2) , is indeed the point of absolute minimum. For this , we should compare the value of $T(x)$ at the point satisfying (1) with the values:

$$\begin{cases} T(0) = \frac{a}{v_1} + \frac{1}{v_2}((d)^2 + b^2)^{\frac{1}{2}} \\ \\ T(d) = \frac{1}{v_1}(a^2 + d^2)^{\frac{1}{2}} + \frac{b}{v_2} \end{cases}$$

But , comparing these three values is not easy because the value of T at x satisfying (1) cannot be calculated easily. Therefore, we come up with another trick: The value of $T'(x)$ over the interval $0 \leq x \leq d$ is only zero at the point x satisfying (1) which for now we call x^*, therefore on the interval $[0, x^*)$ the sign of $T\,'(x)$ does not change , meaning that it is positive or negative everywhere there. To check the sign of $T'(x)$ over that interval we just need to check its sign at a particular point: for example we check it for $x = 0$: we have

$$T'(0) = -\frac{d}{v_2\sqrt{b^2+d^2}} < 0 \quad \Rightarrow \quad T'(x) < 0 \text{ for all } x \in [0\,,\, x^*)$$

In a similar fashion:

$$T'(d) = \frac{d}{v_1\sqrt{a^2+d^2}} > 0 \quad \Rightarrow \quad T'(x) > 0 \text{ for all } x \in (x^*\,,\, d]$$

Therefore we have the following table:

		x^*	
T'	$-$		$+$
T	$\searrow$		$\nearrow$
		absolute min	

Note. Exercise 50 of section 4.7 is basically a similar question and its solution will be in the same lines (its solution is left to the students):

Section 4.7 exercise 50. An underground pipeline is to be constructed between two cities A and B. ana analysis of the substructure indicates that construction costs per kilometer in region I is c_1 and for that of region II is c_2. Show that the total construction cost is minimized when x is chosen so that $c_1 \sin\theta_1 = c_2 \sin\theta_2$

Section 4.7 exercise 53 modified. The cost of fuel per hour for running a ship varies directly as the cube of the speed, and is $B = 100$ dollars per hour when the speed is $s = 40$ kilometers per hour. There are also fixed costs of $A = 200$ dollars per hour. Find the most economical speed at which to make a trip of 1000 kilometers.

Solution. The variable cost is $k\,s^3$ where s is the speed. At $s = 40$ this should be 100. Therefore

$k(40)^3 = 100 \quad \Rightarrow \quad k = \frac{100}{64000} = \frac{1}{640}$

So , the variable cost per hour is $\frac{s^3}{640}$. Then the total cost per hour is

$200 + \frac{s^3}{640} \qquad 0 < s < \infty$

Since it takes $\frac{1000}{s}$ hours to complete the trip, the total cost for this trip is

$C(s) = \frac{1000}{s}\left(200 + \frac{s^3}{640}\right) = \frac{200000}{s} + \frac{25}{16}\,s^2 \qquad 0 < s < \infty$

We want to have the minimum cost , therefore we need to find the absolute minimum of $C(s)$. For this we follow the following steps:

$C'(s) = -\frac{200000}{s^2} + \frac{25}{8}s$

$C'(s) = 0 \quad \Rightarrow \quad \frac{200000}{s^2} = \frac{25}{8}s \quad \Rightarrow \quad s^3 = \frac{(200000)(8)}{25} = 64000$

$\Rightarrow \quad s = 40$

Also note that

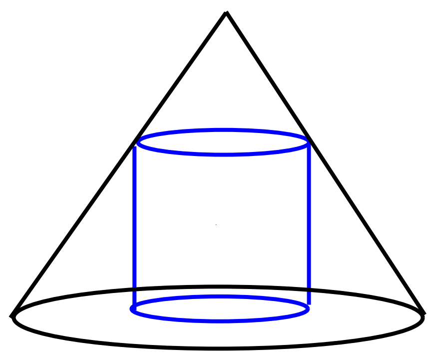
$$\lim_{s\to 0^+} C(s) = +\infty$$

$$\lim_{s\to\infty} C(s) = +\infty$$

therefore we have the following table:

	candidate s	$C(s)$ or $\lim C(s)$	
	40	7500	**absolute min**
(not a candidate)	0	$+\infty$	
(not a candidate)	∞	$+\infty$	

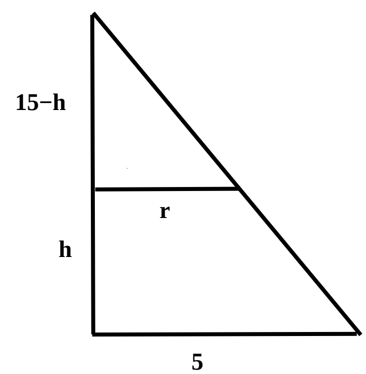
Section 4.7 exercise 72. A right circular cone has radius 5 and height 15. A right circular cylinder is inscribed inside the cone so that its upper edge is on the cone. Find the radius of the cylinder in order that its surface area (including top, bottom, and side) be as large as possible.



Solution. Let the height of the cylinder be called h and its base's radius be called r. These quantities are variable. Let the area of the cylinder be denoted by A. Then

$$A = 2\pi r^2 + 2\pi rh$$

But , this is a function of two unknowns h nd r ; we must remove one of them in order to convert A to a function of one variable only. For this, look at the following triangle cut from the :



From the similarity between triangles, we have

$$\frac{r}{5} = \frac{15-h}{15} \quad \Rightarrow \quad 15 - h = 3r \quad \Rightarrow \quad h = 15 - 3r$$

Substituting this value for h in the above equality , we will have A as a function of r:

$A(r) = 2\pi r^2 + 2\pi r(15 - 3r) = 2\pi r^2 + 30\pi r - 6\pi r^2$

$A(r) = 30\pi r - 4\pi r^2 \qquad 0 < r < 5$

Then

$A'(r) = 30\pi - 8\pi r$

$A'(r) = 0 \quad \Rightarrow \quad 30\pi - 8\pi r = 0 \quad \Rightarrow \quad r = \frac{30}{8} \quad \Rightarrow \quad A(\frac{30}{8}) = \frac{900\pi}{16}$

We also have:

$$\begin{cases} \lim_{r\to 0^+} A(r) = 0 \\ \lim_{r\to 5^-} A(r) = 50\pi \end{cases}$$

	candidate r	$A(r)$ or $\lim A(r)$	
	$\frac{30}{8}$	$\frac{900\pi}{16}$	**absolute max**
(not a candidate)	0	0	
(not a candidate)	5	50π	

So the required radius is $r = \frac{30}{8}$