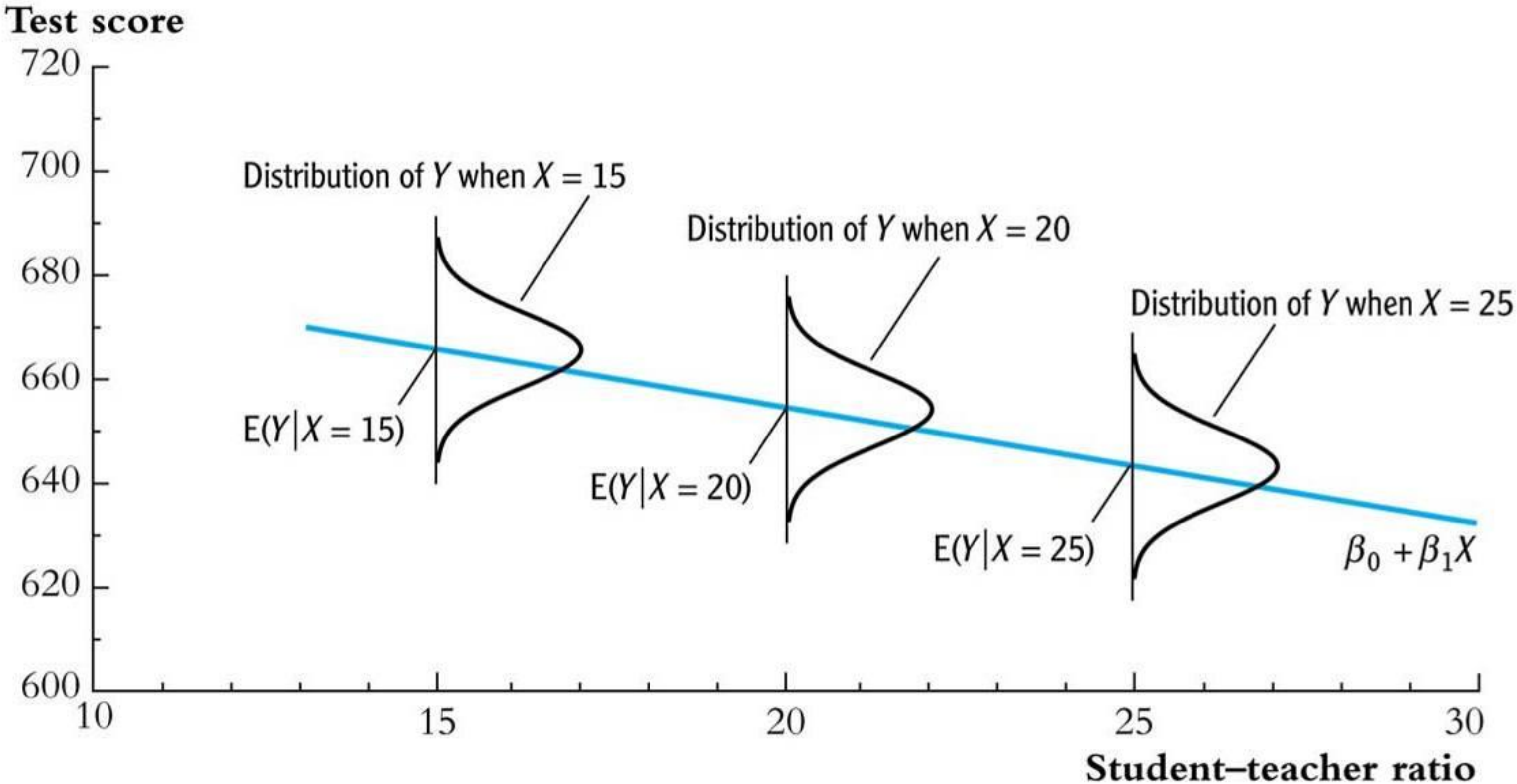
Heteroskedasticity and Homoskedasticity, and Homoskedasticity-Only Standard Errors (Section 5.4)

- What…?
- Consequences of homoskedasticity
- Implication for computing standard errors

What do these two terms mean?

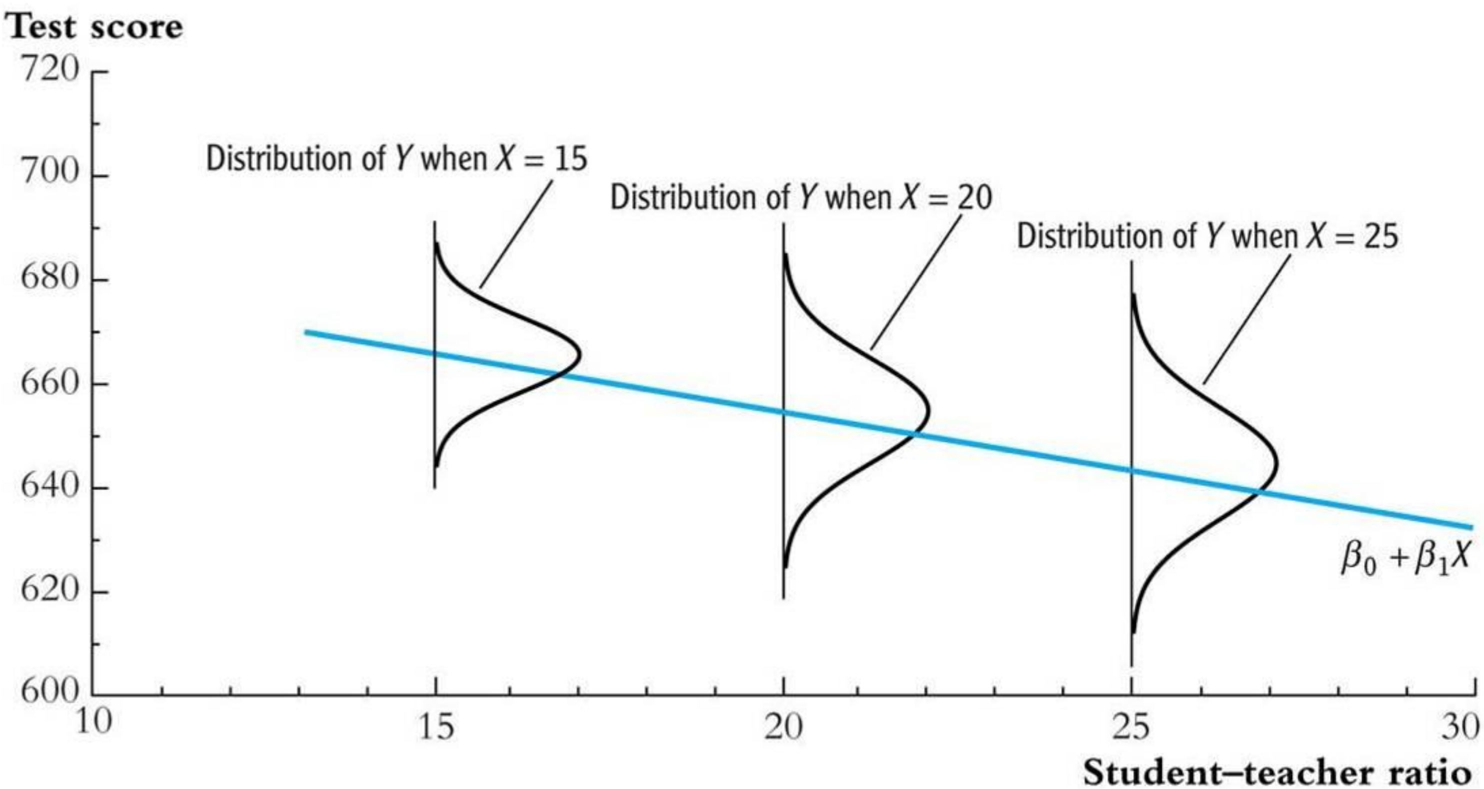
If var($u|X=x$) is constant – that is, if the variance of the conditional distribution of u given X does not depend on X – then u is said to be ***homoskedastic***. Otherwise, u is ***heteroskedastic***.

Homoskedasticity in a picture:



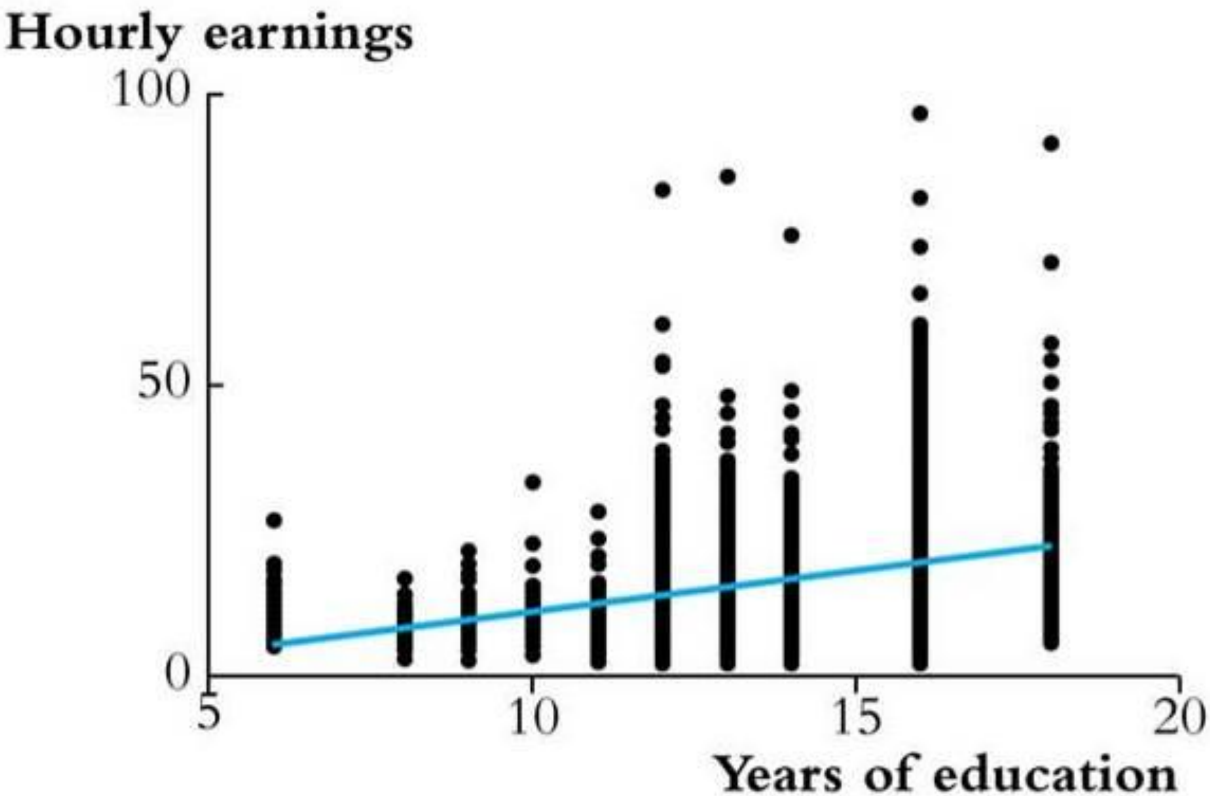
- $E(u|X=x) = 0$ (u satisfies Least Squares Assumption #1)
- The variance of u ***does not*** depend on x

Heteroskedasticity in a picture:



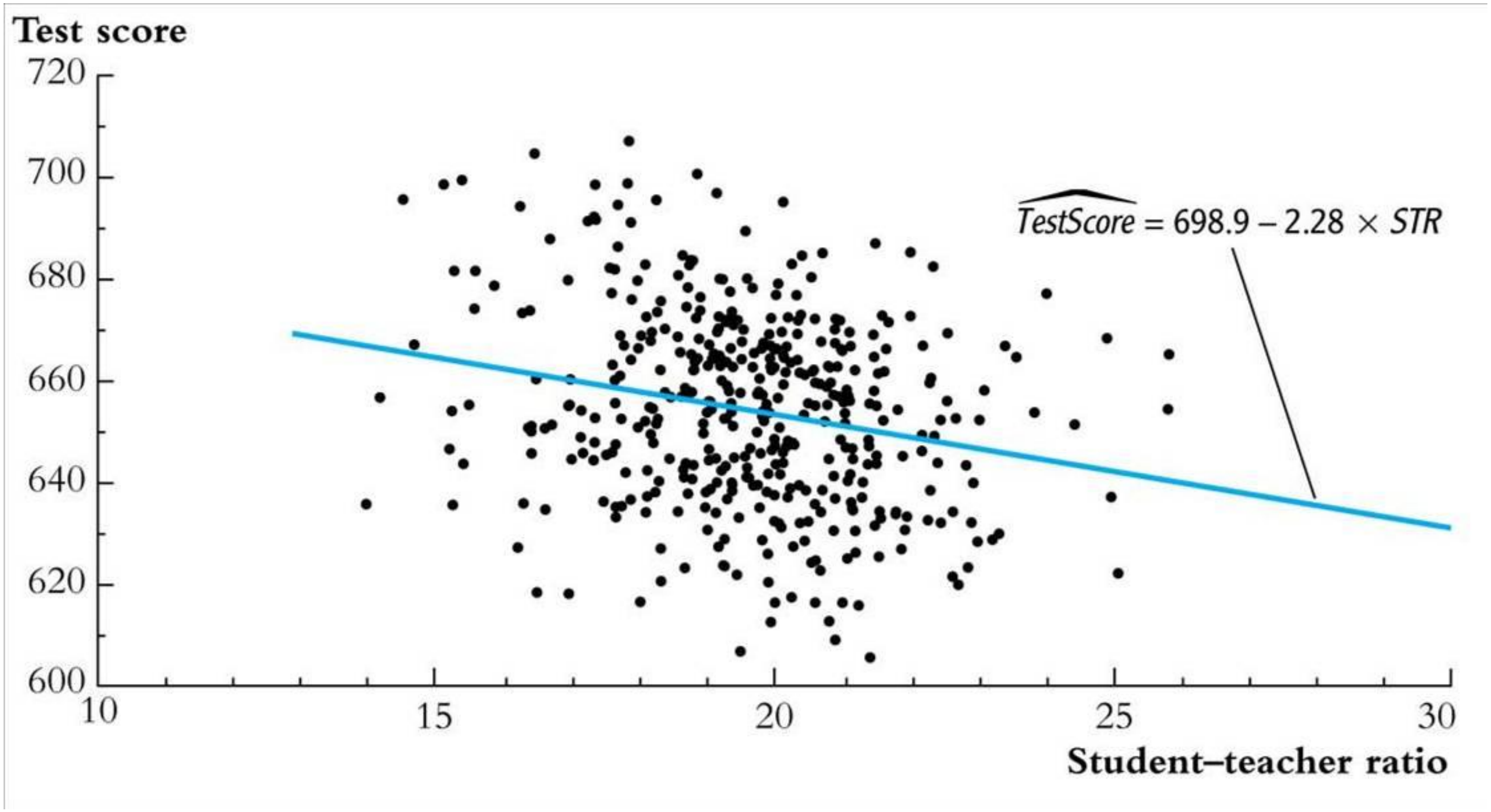
- $E(u|X=x) = 0$ (u satisfies Least Squares Assumption #1)
- The variance of u ***does*** depends on x: u is heteroskedastic.

A real-data example from labor economics: average hourly earnings vs. years of education (data source: Current Population Survey):



Heteroskedastic or homoskedastic?

The class size data:



Heteroskedastic or homoskedastic?

So far we have (without saying so) assumed that *u* might be heteroskedastic.

Recall the three least squares assumptions:

1. $E(u|X = x) = 0$
2. (X_i,Y_i), $i = 1,\ldots,n$, are i.i.d.
3. Large outliers are rare

Heteroskedasticity and homoskedasticity concern $\text{var}(u|X=x)$. Because we have not explicitly assumed homoskedastic errors, we have implicitly allowed for heteroskedasticity.

We now have two formulas for standard errors for $\hat{\beta}_1$

- ***Homoskedasticity-only standard errors*** – these are valid only if the errors are homoskedastic.
- The usual standard errors – to differentiate the two, it is conventional to call these ***heteroskedasticity – robust standard errors***, because they are valid whether or not the errors are heteroskedastic.
- The main advantage of the homoskedasticity-only standard errors is that the formula is simpler. But the disadvantage is that the formula is only correct in general if the errors are homoskedastic.

Practical implications…

- The homoskedasticity-only formula for the standard error of $\hat{\beta}_1$ and the "heteroskedasticity-robust" formula differ – so in general, *you get different standard errors using the different formulas*.
- Homoskedasticity-only standard errors are the default setting in regression software – sometimes the only setting (e.g. Excel). To get the general "heteroskedasticity-robust" standard errors you must override the default.

If you don't override the default and there is in fact heteroskedasticity, your standard errors (and wrong *t*-statistics and confidence intervals) will be wrong – typically, homoskedasticity-only *SE*s are too small.

Heteroskedasticity-robust standard errors in STATA

```
regress testscr str, robust

Regression with robust standard errors               Number of obs =     420
                                                     F(  1,   418) =   19.26
                                                     Prob > F      =  0.0000
                                                     R-squared     =  0.0512
                                                     Root MSE      =  18.581
-------------------------------------------------------------------------
        |               Robust
testscr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
--------+----------------------------------------------------------------
    str |  -2.279808   .5194892    -4.39   0.000    -3.300945   -1.258671
  _cons |    698.933   10.36436    67.44   0.000     678.5602    719.3057
-------------------------------------------------------------------------
```

- If you use the "**, robust**" option, STATA computes heteroskedasticity-robust standard errors
- Otherwise, STATA computes homoskedasticity-only standard errors

The bottom line:

- If the errors are either homoskedastic or heteroskedastic and you use heteroskedastic-robust standard errors, you are OK
- If the errors are heteroskedastic and you use the homoskedasticity-only formula for standard errors, your standard errors will be wrong (the homoskedasticity-only estimator of the variance of $\hat{\beta}_1$ is inconsistent if there is heteroskedasticity).
- The two formulas coincide (when n is large) in the special case of homoskedasticity
- So, you should always use heteroskedasticity-robust standard errors.

Some Additional Theoretical Foundations of OLS (Section 5.5)

We have already learned a very great deal about OLS: OLS is unbiased and consistent; we have a formula for heteroskedasticity-robust standard errors; and we can construct confidence intervals and test statistics.

Also, a very good reason to use OLS is that everyone else does – so by using it, others will understand what you are doing. In effect, OLS is the language of regression analysis, and if you use a different estimator, you will be speaking a different language.

The Extended Least Squares Assumptions

These consist of the three LS assumptions, plus two more:

1. $E(u|X = x) = 0$.
2. (X_i,Y_i), $i = 1,\ldots,n$, are i.i.d.
3. Large outliers are rare ($E(Y^4) < \infty$, $E(X^4) < \infty$).
4. u is homoskedastic
5. u is distributed $N(0,\sigma^2)$

- Assumptions 4 and 5 are more restrictive – so they apply to fewer cases in practice. However, if you make these assumptions, then certain mathematical calculations simplify and you can prove strong results – results that hold if these additional assumptions are true.
- We start with a discussion of the efficiency of OLS

Efficiency of OLS, part I: The Gauss-Markov Theorem

Under extended LS assumptions 1-4 (the basic three, plus homoskedasticity), $\hat{\beta}_1$ has the smallest variance among *all linear estimators* (estimators that are linear functions of $Y_1,\ldots, Y_n$). This is the ***Gauss-Markov theorem***.

Comments

- The GM theorem is proven in SW Appendix 5.2

Efficiency of OLS, part II:

- Under <u>all five</u> extended LS assumptions – including normally distributed errors – $\hat{\beta}_1$ has the smallest variance of <u>*all*</u> consistent estimators (linear *or* nonlinear functions of $Y_1,\ldots,Y_n$), as $n \to \infty$.
- This is a pretty amazing result – it says that, if (in addition to LSA 1-3) the errors are homoskedastic and normally distributed, then OLS is a better choice than any other consistent estimator. And because an estimator that isn't consistent is a poor choice, this says that OLS really is the best you can do – if all five extended LS assumptions hold. (The proof of this result is beyond the scope of this course and isn't in SW – it is typically done in graduate courses.)

Some not-so-good thing about OLS

The foregoing results are impressive, but these results – and the OLS estimator – have important limitations.

1. The GM theorem really isn't that compelling:
   - The condition of homoskedasticity often doesn't hold (homoskedasticity is special)
   - The result is only for linear estimators – only a small subset of estimators (more on this in a moment)

2. The strongest optimality result ("part II" above) requires homoskedastic normal errors – not plausible in applications (think about the hourly earnings data!)

Limitations of OLS, ctd.

3. OLS is more sensitive to outliers than some other estimators. In the case of estimating the population mean, if there are big outliers, then the median is preferred to the mean because the median is less sensitive to outliers – it has a smaller variance than OLS when there are outliers. Similarly, in regression, OLS can be sensitive to outliers, and if there are big outliers other estimators can be more efficient (have a smaller variance). One such estimator is the least absolute deviations (LAD) estimator:

$$\min_{b_0, b_1} \sum_{i=1}^{n} \left| Y_i - (b_0 + b_1 X_i) \right|$$

In virtually all applied regression analysis, OLS is used – and that is what we will do in this course too.

Summary and Assessment

(Section 5.7)

- The initial policy question:

  Suppose new teachers are hired so the student-teacher ratio falls by one student per class. What is the effect of this policy intervention ("treatment") on test scores?

- Does our regression analysis answer this convincingly?
  - *Not really* – districts with low *STR* tend to be ones with lots of other resources and higher income families, which provide kids with more learning opportunities outside school…this suggests that corr(u_i, STR_i) > 0, so $E(u_i|X_i) \neq 0$.
  - So, we have omitted some factors, or variables, from our analysis, and this has biased our results.