Assignment #4 Answer Key

Q.1. a) $L = p(y_1, \ldots, y_n | \beta, \sigma)$

$$= \prod_{i=1}^{n} p(y_i | \beta, \sigma) \quad ; \text{ given independence}$$

$$= \prod_{i=1}^{n} \left[\frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-1}{2\sigma^2}(y_i - x_i'\beta)^2} \right]$$

$$= \frac{1}{\sigma^n (\sqrt{2\pi})^n} \cdot e^{\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (y_i - x_i'\beta)^2}$$

$$= \sigma^{-n} (2\pi)^{-n/2} e^{\left(\frac{1}{2\sigma^2} \sum_{i=1}^{n} [y_i - x_i'\beta]^2\right)}$$

b) $\log L = -n \log\sigma - \frac{n}{2} \log 2\pi - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - x_i'\beta)^2$

$$= -n \log\sigma - n/2 \log 2\pi - 1/2\sigma^2 (y - X\beta)'(y - X\beta)$$

$$\frac{\partial \log L}{\partial \beta} = \frac{-1}{2\sigma^2} \cdot \frac{\partial}{\partial \beta} \left[y'y + \beta'X'X\beta - 2y'X\beta \right]$$

$$= \frac{-1}{2\sigma^2} (2X'X\beta - 2X'y) = 0$$

Hence, $(X'X\beta - X'y) = 0 \quad \Rightarrow \hat{\beta} = (X'X)^{-1}X'y$,

and this is just the OLS estimator, which is unbiased under the usual set of assumptions.

c) With OLS, the derivation of b requires:

(i) Linear model;

(ii) X has full rank.

(the properties of b then depend on other assumptions).

With the MLE, the derivation of the estimator requires (i) and (ii) above, and an appropriate assumption about the distribution of the errors - here, normality was used. The properties of the MLE are the same as those of b, if the same assumptions hold.

d) $\dfrac{\partial \log L}{\partial \sigma^2} = \dfrac{\partial}{\partial \sigma^2}\left[-\frac{n}{2}\log\sigma^2 - \frac{n}{2}\log 2\pi - \frac{1}{2\sigma^2}(y-X\beta)'(y-X\beta)\right]$

$= -\dfrac{n}{2\sigma^2} + \dfrac{1}{2\sigma^4}(y-X\beta)'(y-X\beta) = 0$

So, $\tilde{\sigma}^2 = \dfrac{1}{n}(y-X\tilde{\beta})'(y-X\tilde{\beta}) = \frac{1}{n}e'e$,

where e = MLE residual vector = OLS residual vector

Compare this with $s^2 = \dfrac{1}{n-K}e'e$, our usual estimator. We Know that s^2 is unbiased, so clearly $\hat{\sigma}^2$ is biased.

e) $E(\hat{\sigma}^2) = \frac{1}{n}E(e'e) = \frac{1}{n}(n-k)\sigma^2 = (1-K/n)\sigma^2 \neq \sigma^2$ (Biased)

and $\text{Bias}(\hat{\sigma}^2) = E(\hat{\sigma}^2) - \sigma^2 = (1-K/n)\sigma^2 - \sigma^2 = (-K/n)\sigma^2$.

So, Bias is negative. For fixed K, if $n \to \infty$, then this bias vanishes.