2.2)

a) b is just the OLS estimator of β. For the vector case, its covariance matrix is $\sigma^2 (X'X)^{-1}$, so here we should have:

$$\mathrm{var}(b) = \sigma^2 / \sum_i x_i^2 .$$ To check this:

$$\mathrm{var}(b) = \mathrm{var}\left[(\Sigma x_i y_i) / (\Sigma x_i^2)\right] = \left[\frac{1}{\Sigma x_i^2}\right]^2 \mathrm{var}\left[\Sigma x_i y_i\right].$$

Now, the x_i's are non-random, so

$$\mathrm{var}(b) = \left[\frac{1}{\Sigma x_i^2}\right]^2 \Sigma\, \mathrm{var}(x_i y_i)$$

$$= \left[\frac{1}{\Sigma x_i^2}\right]^2 \Sigma x_i^2\, \mathrm{var}(y_i),$$

where we have used the fact that the y_i's are uncorrelated, so $\mathrm{var}\, \Sigma y_i = \Sigma\, \mathrm{var}(y_i)$.

Finally, $\mathrm{var}(b) = \left[\frac{1}{\Sigma x_i^2}\right]^2 (\Sigma x_i^2) \sigma^2 = \sigma^2 / \Sigma x_i^2$ #

(b) $E(\tilde{\beta}) = E\{\bar{Y}/\bar{X}\} = \left(\frac{1}{\bar{X}}\right)E(\bar{Y}) = \frac{1}{\bar{X}}E\left[\frac{1}{n}\Sigma(X_i\beta + \varepsilon_i)\right]$

$$= \left(\frac{1}{n\bar{X}}\right)\Sigma\left[X_i\beta + E(\varepsilon_i)\right]$$

$$= \left(\frac{1}{n\bar{X}}\right)\Sigma X_i\beta = \left(\frac{n\bar{X}}{n\bar{X}}\right)\beta = \beta \quad \#$$

This estimator is unbiased.

$$E(\hat{\beta}) = E\left[\frac{1}{n}\Sigma(Y_i/X_i)\right] = \frac{1}{n}\Sigma\left(\frac{1}{X_i}\right)E(Y_i)$$

$$= 1/n\,\Sigma(1/X_i)\,E(X_i\beta + \varepsilon_i) = 1/n\,\Sigma\frac{1}{X_i}(X_i\beta + 0)$$

$$= 1/n\,\Sigma\beta = \frac{1}{n}n\beta = \beta \quad \#$$

This estimator is also unbiased. (All 3 estimators are unbiased).

Note, also, that

$$\tilde{\beta} = \left(\frac{1}{n}\Sigma Y_i\right)/\left(\frac{1}{n}\Sigma X_i\right) = \left(\Sigma Y_i/\Sigma X_i\right) = \Sigma w_i Y_i$$

where $w_i = \frac{1}{\Sigma X_i} = \frac{1}{n\bar{X}}$. (Fixed weights). This is a linear combination of the Y_i's.

Finally, $\hat{\beta} = \frac{1}{n}\Sigma(Y_i/X_i) = \Sigma\left(\frac{1}{nX_i}\right)Y_i = \Sigma a_i Y_i$

where $a_i = \frac{1}{nX_i}$, and these weights are also non-random.

$\hat{\beta}$ is also a linear estimator.

c) Both $\tilde{\beta}$ and $\hat{\beta}$ are linear combinations of the y_i's, and the latter are Normal if the errors are Normal. Linear combinations of Normal random variables are Normal, so both $\tilde{\beta}$ and $\hat{\beta}$ have Normal sampling distributions.

d) Recall from class that the fitted OLS line passes through $(\bar{x}, \bar{y})$, if the model has an intercept. However, in our case –

$$\hat{y}_j = x_j b,$$

So, when $x_j = \bar{x}$:

$$y_j = \bar{x}\left[\Sigma x_i y_i / \Sigma x_i^2\right] \neq \bar{y}, \text{ in general.}$$

e.g. if $n=2$, $y = \{4, 8\}$; $x = \{1, 3\}$, then $\Sigma x_i y_i = 28$;

$\Sigma x_i^2 = 10$; $b = 2.8$; $\bar{x} = 2$ and $\bar{y} = 6$. However, $\hat{y} = 5.6$ when $x = \bar{x}$.

Consider $\tilde{\beta}$:

When $x_i = \bar{x}$, $\hat{y}_i = \bar{x}\left[\bar{y}/\bar{x}\right] = \bar{y}$ #

This line passes through $(\bar{x}, \bar{y})$.

Finally, consider $\hat{\beta}$:

$\hat{\beta} = \frac{1}{n}\sum_i (y_i/x_i)$, and when we fit through $\bar{x}$:

$$\hat{y}_i = \bar{x}\frac{1}{n}\sum_i (y_i/x_i) \neq \bar{y}, \text{ in general.}$$

e) We know that $\mathrm{var}(b) = \sigma^2 / \Sigma x_i^2$.

$$\mathrm{var}(\hat{\beta}) = \mathrm{var}\left[\tfrac{1}{n} \Sigma (y_i / x_i)\right] = \frac{1}{n^2} \mathrm{var}\, \Sigma (y_i / x_i)$$

$$= \frac{1}{n^2} \Sigma\, \mathrm{var}(y_i / x_i) = \tfrac{1}{n^2} \Sigma \left(\frac{1}{x_i^2}\right) \mathrm{var}(y_i)$$

$$= \tfrac{1}{n^2} \Sigma\, \sigma^2 / x_i^2 = \frac{\sigma^2}{n^2} \Sigma \left(\frac{1}{x_i^2}\right) \quad \#$$

$$\mathrm{var}(\tilde{\beta}) = \mathrm{var}[\bar{y} / \bar{x}] = \frac{1}{\bar{x}^2} \mathrm{var}(\bar{y}) = \frac{1}{\bar{x}^2} \mathrm{var}\left[\tfrac{1}{n} \Sigma y_i\right]$$

$$= \left(\frac{1}{\bar{x}^2}\right)\left(\frac{1}{n^2}\right) \mathrm{var}\, \Sigma (y_i) = \frac{1}{n^2 \bar{x}^2} \Sigma\, \mathrm{var}(y_i)$$

$$= \frac{n \sigma^2}{n^2 \bar{x}^2} = \frac{\sigma^2}{n \bar{x}^2} \quad \#$$

All 3 estimators are linear and unbiased. So, by the Gauss-Markhov Theorem, b is "best." That is, $\mathrm{var}(b) \le \mathrm{var}(\hat{\beta})$ and $\mathrm{var}(b) \le \mathrm{var}(\tilde{\beta})$. Can we say more than this? First, even if we did not know about the Gauss-Markhov Theorem, we can show that $\mathrm{var}(b) \le \mathrm{var}(\hat{\beta})$, for example, by using the Cauchy-Schwarz inequality. This inequality can be written in various forms, but here the useful form is:

$$\left(\sum_i a_i b_i\right)^2 \le \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right).$$

Let's check if $\left(\sigma^2 / \sum_i x_i^2\right) \le \sigma^2 / n^2 \sum_i \left(\frac{1}{x_i^2}\right)$

i.e., if $n^2 \le \left(\sum_i x_i^2\right)\left(\sum_i \frac{1}{x_i^2}\right)$.

Let $a_i = x_i$; $b_i = 1/x_i$. Then, the C.S.I. says:

$$\left(\sum_i x_i \tfrac{1}{x_i}\right)^2 \le \left(\sum_i x_i^2\right)\left(\sum_i \left(\tfrac{1}{x_i^2}\right)\right)$$

or $$\left(\Sigma 1\right)^2 \le \left(\Sigma x_i^2\right)\left(\sum_i \left(\tfrac{1}{x_i^2}\right)\right)$$

or $$n^2 \le \left(\Sigma x_i^2\right)\left(\Sigma 1/x_i^2\right) \quad \checkmark \qquad \text{(as required)}$$

To show that $\mathrm{var}(b) \le \mathrm{var}(\tilde{\beta})$, we need to show that $\cancel{\sigma^2/\Sigma x_i^2}$

$$\sigma^2/\Sigma x_i^2 \le \frac{\sigma^2}{n\bar{x}^2}, \quad \text{or that } \left(\Sigma x_i\right)^2 \le n \Sigma x_i^2.$$

Let $a_i = 1$ and $b_i = x_i$ and it follows that this holds, by the C.S.I.

The C.S.I. doesn't help in this way when we try to compare $\mathrm{var}(\hat{\beta})$ and $\mathrm{var}(\tilde{\beta})$. In fact the efficiency can go either way here, depending on the x_i values. For example, let $n = 2$. Then:

(i) If $x_1 = -1$, $x_2 = 0.9$; $\mathrm{var}(\hat{\beta}) = \frac{1}{4}\left[\frac{1}{(-1)^2} + \frac{1}{(0.9)^2}\right] = 0.559$,

and $\mathrm{var}(\tilde{\beta}) = 2/[-1 + 0.9]^2 = 200$. So $\mathrm{var}(\hat{\beta}) < \mathrm{var}(\tilde{\beta})$.

(ii) If $x_1 = 1/2$, $x_2 = 1$; $\mathrm{var}(\hat{\beta}) = \frac{1}{4}(5) = 1.25$, and $\mathrm{var}(\tilde{\beta}) = 2/1.5^2 = 0.\bar{8}$

So $\mathrm{var}(\tilde{\beta}) < \mathrm{var}(\hat{\beta})$ in this case. The relative efficiency for these two estimators is indeterminate – it is data-dependent.

3) See the EViews file on the course web page.