ECON 7010: Assignment #2 Answer Key

2.1 a) $\hat{\beta} = (A + X'X)^{-1}X'y = Cy$,

where $C = (A + X'X)^{-1}X'$, and C is non-random if X is non-random.

Hence, $\hat{\beta}$ is a linear estimator.

b) A and X are non-random. So, $\text{plim}[\frac{1}{n}X'\varepsilon] = 0$, and $\text{plim}[\frac{1}{n}A] = \text{limit}[\frac{1}{n}A] = 0$; $\text{plim}[\frac{1}{n}X'X] = \lim[\frac{1}{n}X'X] = Q$

So, $\hat{\beta} = [\frac{1}{n}A + \frac{1}{n}X'X]^{-1}[(\frac{1}{n}X'X\beta) + (\frac{1}{n}X'\varepsilon)]$

and $\text{plim}(\hat{\beta}) = Q^{-1}Q\beta + Q^{-1}\cdot 0 = \beta$ (consistent)

c) $E(\hat{\beta}) = E[(A + X'X)^{-1}X'y] = (A + X'X)^{-1}X'E(y)$

So, $E(\hat{\beta}) = (A + X'X)^{-1}X'E(X\beta + \varepsilon)$
$= (A + X'X)^{-1}X'X\beta$ (if the errors have zero mean)

$\text{Bias}(\hat{\beta}) = E(\hat{\beta}) - \beta = (A + X'X)^{-1}X'X\beta - \beta$

d) The modified estimator is

$$\beta^* = \hat{\beta} - \text{Bias}(\hat{\beta}) = (A + X'X)^{-1}X'y - (A + X'X)^{-1}X'X\beta + \beta$$

$$= (A + X'X)^{-1}X'(y - X\beta) + \beta,$$

and this depends on the unknown β, so it can't be constructed in practice.

e) Now replace β by b (the OLS estimator of β) in the expression for β^*:

$$\hat{\beta}^* = (A + X'X)^{-1}X'y - (A + X'X)^{-1}X'Xb + b$$

$$= (A + X'X)^{-1}(X'y - X'Xb) + b$$

But $b = (X'X)^{-1}X'y$, so $X'Xb = X'y$, and

$$\hat{\beta}^* = 0 + b = b$$

Q.2. See EViews workfile.

Q.3. a) $X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & n \end{bmatrix}$; $X' = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & n \end{bmatrix}$

$$X'X = \begin{bmatrix} n & \sum_{i=1}^{n} i \\ \sum_{i=1}^{n} i & \sum_{i=1}^{n} i^2 \end{bmatrix} = \begin{matrix} n & \frac{n(n+1)}{2} \\ \frac{n(n+1)}{2} & \frac{n(n+1)(2n+1)}{6} \end{matrix}$$

So, $\left(\frac{1}{n}X'X\right) = \begin{bmatrix} 1 & \frac{(n+1)}{2} \\ \frac{(n+1)}{2} & \frac{(n+1)(2n+1)}{6} \end{bmatrix}$

and $\underset{n\to\infty}{\text{Limit}} \left[\frac{1}{n}X'X\right]$ is a matrix with 3 infinitely large elements. It is not a finite matrix. Its determinant is infinite, and hence positive, so it is still positive definite.

b) $b=(X'X)^{-1}X'y$. Clearly the regressors are non-random, so $E(b)=\beta$, assuming that $E(\varepsilon_i)=0$. So, $E(b_2)=\beta_2$, in particular. If the errors have a scalar covariance matrix, $V(b)=\sigma^2(X'X)^{-1}$.

Here, $$(X'X)=\begin{bmatrix} n & \frac{n(n+1)}{2} \\ \frac{n(n+1)}{2} & \frac{n(n+1)(2n+1)}{6} \end{bmatrix}$$

$$|X'X| = \frac{n^2(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4}$$

$$= \frac{n^2(n-1)(n+1)}{12}$$

So, $$(X'X)^{-1} = \frac{12}{n^2(n-1)(n+1)}\begin{bmatrix} \frac{n(n+1)(2n+1)}{6} & -\frac{n(n+1)}{2} \\ -\frac{n(n+1)}{2} & n \end{bmatrix}$$

and $$\text{var}(b_2) = \frac{12}{n^2(n-1)(n+1)} \cdot n = \frac{12}{n(n-1)(n+1)}$$

c) Clearly, as b_2 is unbiased, and $var(b_2) \to 0$ an $n \to \infty$, b_2 is mean-square consistent, and hence it is also weakly
d) consistent ($plim(b_2) = \beta_2$).

e) $$var(b_1) = \left[\frac{12}{n^2(n-1)(n+1)} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{2(n+1)(2n+1)}{n(n-1)(n+1)} = \frac{4n^2+6n+2}{n^3-n} = \frac{4/n + 6/n^2 + 2/n^3}{1-(1/n^2)}$$

$$\to 0 \quad \text{as} \quad n \to \infty$$

So, b_1 is unbiased and $var(b_1) \to 0$ ~~an~~ as $n \to \infty$, so b_1 is mean-square consistent, and hence also weakly consistent.

f) The usual assumption is sufficient, but not necessary, for weak consistency.

Assignment #3 - Answer Key

1. a) In general, the F-test statistic for testing J linear restrictions can be expressed as:

$$F = \frac{(e^{*\prime}e^{*} - e'e)/J}{e'e/(n-k)}$$

Here, $J = K-1$. What does $e^{*\prime}e^{*}$ look like? When we set all coefficients except for the intercept's to be zero, we are just regressing y on a constant. In this case, the intercept coefficient estimator is $\bar{y}$, and the residuals are of the form $(y_i - \bar{y})$. So, $e^{*\prime}e^{*} = \sum_i (y_i - \bar{y})^2$.

Recall that $R^2 = 1 - \dfrac{e'e}{\sum_i (y_i - \bar{y})^2}$. So,

$$\frac{R^2}{K-1} = \frac{\left[\sum (y_i - \bar{y})^2 - e'e\right]/(K-1)}{\sum (y_i - \bar{y})^2} = \frac{(e^{*\prime}e^{*} - e'e)/(K-1)}{\sum (y_i - \bar{y})^2}$$

and $\dfrac{(1-R^2)}{(n-k)} = \dfrac{e'e/(n-k)}{\sum (y_i - \bar{y})^2}$. So,

$$[R^2/(K-1)] / [(1-R^2)/(n-k)] = \frac{(e^{*\prime}e^{*} - e'e)/(K-1)}{e'e/(n-k)} = F$$

b) Show that $\partial F / \partial R^2 > 0$, always:

$$\partial F / \partial R^2 = \left(\frac{n-k}{K-1}\right) \frac{\partial}{\partial R^2}\left[\frac{R^2}{1-R^2}\right],$$ where $n-k > 0$, and $K-1 > 0$.

$K-1>0$ since neither the F-test nor R^2 should be used unless there is an intercept in the model, so $K \geq 2$.

$$\frac{\partial [R^2/(1-R^2)]}{\partial R^2} = \left(\frac{1}{1-R^2}\right) + R^2(-1)(-1)(1-R^2)^{-2}$$

$$= \frac{1}{1-R^2} + \frac{R^2}{(1-R^2)^2} = \frac{1-R^2+R^2}{(1-R^2)^2} = \frac{1}{(1-R^2)^2} > 0$$

So, there is a positive monotonic relationship between F and R^2. When one rises in value, so must the other.

2. The first model is: $y = X\beta_1 + \alpha d + \varepsilon$
(α is a scalar, since d is a vector).

Using our partitioned regression results:

$\hat{\beta}_1 = (X'M_d X)^{-1}X'M_d y$, ⓐ where $M_d = I - d(d'd)^{-1}d'$

and $d' = (0, 0, \ldots, 1)$. So, $(d'd)^{-1} = 1^{-1} = 1$, and

$$M_d = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix}$$

Now consider the situation when we drop 'd' from the model and delete the last observation. Define $y_{(n)} = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$ and similarly for $X_{(n)}$.

(The '(n)' denotes the n^{th} row of the vector or matrix is deleted. So, $y_{(n)}$ is $(n-1)\times 1$ and $X_{(n)}$ is $(n-1)\times K$.)