Q1. $var(\varepsilon) = \Sigma$. So, $var(y) = \Sigma$.

So, $var(b) = var[(X'X)^{-1}X'y] = (X'X)^{-1}X' var(y) X (X'X)^{-1}$

$= (X'X)^{-1}X'\Sigma X(X'X)^{-1}$ [Familiar]

$\hat{\beta}_{GLS} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$

$\Sigma^{-1'} = \Sigma^{-1}$

$var(\hat{\beta}_{GLS}) = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}\Sigma\Sigma^{-1'}X(X'\Sigma^{-1}X)^{-1}$

$= (X'\Sigma^{-1}X)^{-1}$ [Familiar]

Now, if $\Sigma X = X\Theta$, then

$var(b) = (X'X)^{-1}X'X\Theta(X'X)^{-1} = \Theta(X'X)^{-1}$

Also,

$\Sigma X = X\Theta$

$X = \Sigma^{-1}X\Theta$

$X'X = X'\Sigma^{-1}X\Theta$

$(X'X)\Theta^{-1} = X'\Sigma^{-1}X$

$\Theta(X'X)^{-1} = (X'\Sigma^{-1}X)^{-1}$

So, $var(b) = var(\hat{\beta}_{GLS})$

$V(b) = (X'X)^{-1} X'X\Theta (X'X)^{-1} = \Theta (X'X)^{-1}$

Also, $X = \Sigma^{-1} X \Theta$

$X'X = X'\Sigma^{-1} X\Theta$ and $(X'X)\Theta^{-1} = X'\Sigma^{-1}X$. So,

$(X'\Sigma^{-1}X)^{-1} = \Theta (X'X)^{-1}$, and so $V(b) = V(\hat{\beta}_{GLS})$

5. (a) Yes, it is inefficient. What invalidates the tests is the fact that the wrong covariance matrix formula is being used for b. Also, inconsistent covariance matrix estimator.

(b) It is unbiased and consistent. The inconsistency arises with the estimation of $V(b)$, not b itself.

(c) No, the transformation is to get an error term with a constant variance.

(d) The log-transformation will "compress" differences in variability. To then fit the log-model with an additive error term we must have been happy with the idea that the original linear specification had a multiplicative error term. So, if taking logs truly eliminates heteroskedasticity in this sense, our original linear model was mis-specified.